*International Journal of Theoretical Physics, Vol. 40, No. 3, 2001*

# Fields on the Poincaré Group: Arbitrary Spin Description and Relativistic Wave Equations

**D. M. Gitman[1,3] and A. L. Shelepin[1,2]**

*Received August 25, 2000*

In this paper, starting from a pure group-theoretic point of view, we develop an approach to describing particles with different spins in the framework of a theory of scalar fields on the Poincaré group. Such fields can be considered as generating functions for conventional spin-tensor fields. The case of two, three, and four dimensions are elaborated in detail. Discrete transformations $C$, $P$, $T$ are defined for the scalar fields as automorphisms of the Poincaré group. We classify the scalar functions, and obtain relativistic wave equations for particles with definite spin and mass. There exist two different types of scalar functions (which describe the same mass and spin), one related to a finite-dimensional nonunitary representation and the other to an infinite-dimensional unitary representation of the Lorentz subgroup. This allows us to derive both usual finite-component wave equations for spin-tensor fields and positive-energy, infinite-component wave equations.

## 1. INTRODUCTION

Traditionally in field theory, particles with different spins are described by multicomponent spin-tensor fields on Minkowski space. However, it is possible to use for this purpose scalar functions as well, which depend on both Minkowski space coordinates and on continuous bosonic variables corresponding to spin degrees of freedom. Such fields were introduced (Bargmann and Wigner, 1948; Ginzburg and Tamm, 1947; Shirokov, 1951; Yukawa, 1950) in connection with the problem of constructing relativistic wave equations (RWE). Fields of this type may be treated as fields on homogeneous spaces of the Poincaré group. A systematic development of this point of view was given by Finkelstein (1955). He also gave a classification and explicit constructions of homogeneous spaces of the Poincaré group, which contain Minkowski space. The next logical step was

[1] Instituto de Física, Universidade de São Paulo, 05315-970-São Paulo, SP, Brazil.
[2] Moscow Institute of Radio Engenering, Electronics and Automation, 117454, Moscoẁ, Russia.
[3] To whom correspondence should be addressed at Instituto de Física, Universidade de São Paulo, Caixa Postal 66318, 05315-970-São Paulo, SP, Brazil; e-mail: gitman@fma.if.usp.br



0020-7748/01/0300-0603$19.50/0 

taken by Lurçat (1964), who suggested constructing quantum field theory on the Poincaré group. One of the motivations was to give a dynamical role to the spin. These ideas were developed in Arodź (1976), Bacry and Kihlberg (1969), Boyer and Fleming (1974), Drechsler (1997), Kihlberg (1970), and Toller (1978, 1996). For example, different homogeneous spaces were described, as well as possibilities to introduce interactions in spin phase space and to construct Lagrangian formulations. Bacry and Kihlberg (1969) concluded that eight is the lowest dimension of a homogeneous space suitable for a description of both half-integer and integer spins. However, no convincing physical motivation for the choice of homogeneous spaces was presented, and the interpretation of additional degrees of freedom and of corresponding quantum numbers remained an open problem.

In this paper, starting from a pure group-theoretic point of view, we develop a regular approach to describing particles with different spins in the framework of a theory of scalar fields on the Poincaré group. Such fields can be considered as generating functions for conventional spin-tensor fields. In this language, the problem of constructing RWE of different types is formulated from a unique position.

We use scalar fields on the proper Poincaré group, that is, fields on the 10-dimensional manifold; this manifold is a direct product of Minkowski space and of the manifold of the Lorentz subgroup. These fields arise in our constructions in the course of studying a generalized regular representation (GRR). This provides the possibility to analyze all the representations of the Poincaré group. Study of a GRR implies the use of harmonic analysis (Barut and Raczka, 1977; Vilenkin, 1968; Vilenkin and Klimyk, 1991; Zhelobenko and Schtern, 1983). In a sense, this method is an alternative to that of induced representations suggested by Wigner (1939) (also see Barut and Raczka, 1977; Kim and Noz, 1986; Mackey, 1968; Ohnuki, 1988). It turns out that the fields on the Poincaré group can be considered as generating functions for the usual spin-tensor fields on Minkowski space, and thus we naturally obtain all results for the latter fields. However, sometimes it is more convenient to formulate properties and equations for spin-tensor fields in terms of the generating functions. Moreover, the problem of constructing RWE is very natural in the language of the scalar fields on the group. We show that this problem can be formulated as a problem of classifying different scalar fields. For this purpose, in accordance with the general theory of harmonic analysis, we consider various sets of commuting operators and identify constructing RWE with eigenvalue problems for these operators. We define discrete transformations for the scalar fields using automorphisms of the proper Poincaré group. The space of scalar fields on the group turns out to be closed with respect to the discrete transformations. The latter transformations are of fundamental importance for constructing RWE and for their analysis. Consideration of the discrete transformations helps us to give the right physical interpretation for quantum numbers that appear in course of classifying the scalar fields.

The paper is organized as follows. In Section 2, we introduce the basic objects of study, namely, scalar fields $f(x, \mathbf{z})$. The scalar fields depend on $x$, which are coordinates on Minkowski space, and on $\mathbf{z}$, which are coordinates on the Lorentz subgroup. The complex coordinates $\mathbf{z}$ describe spin degrees of freedom. It is shown that these fields are generating functions for the usual spin-tensor fields. Classifying the scalar fields with the help of various sets of commuting operators on the group, we get a description of irreps of the group. We formulate a general scheme of constructing RWE in this language in any number of dimensions. We introduce discrete transformations in the space of the scalar functions, and we relate these transformations to automorphisms of the proper Poincaré group.

In Section 3, we apply the above general scheme to a detailed study of scalar fields on two-dimensional Poincaré and Euclidean groups. In particular, we construct RWE and analyze their solutions.

The three-dimensional case of Poincaré and Euclidean groups is considered in Section 4. Besides finite-component equations, we also construct positive-energy RWE associated with unitary infinite-dimensional irreps of the $2+1$ Lorentz group. These equations, in particular, describe particles with fractional spins.

In Section 5, we study scalar fields on the $3+1$ proper Poincaré group. The connection of the present consideration with other approaches to RWE theory is considered in detail. In particular, we consider equations with subsidiary conditions. General first-order Gel'fand–Yaglom equations (including Bhabha equations), Dirac–Fierz–Pauli equations, and Rarita–Schwinger equations arise in the present consideration as well. This gives a basis for comparison of properties of various RWE.

Classifying scalar functions in two, three, and four dimensions, we obtain equations describing fields with fixed mass and spin. In Section 6, we consider the general features of these equations.

The construction of RWE is elaborated in detail only for the massive case. We plan to discuss the massless case in a later article.

## 2. FIELDS ON THE PROPER POINCARÉ GROUP AND SPIN DESCRIPTION

### 2.1. Parametrization of the Poincaré Group

Consider Poincaré group transformations

$$x'^{\nu} = \Lambda^{\nu}_{\mu} x^{\mu} + a^{\nu} \tag{2.1}$$

of coordinates $x = (x^{\mu}, \mu = 0, \ldots, D)$ in $d = (D+1)$-dimensional Minkowski space, $ds^2 = \eta_{\mu\nu}\, dx^{\mu}\, dx^{\nu}$, $\eta_{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$. The matrices $\Lambda$ define rotations in Minkovski space and belong to the vector representation of the $O(D, 1)$ group. We are also going to consider the $D$-dimensional Euclidean case in which

$ds^2 = \eta_{ik}\, dx^i\, dx^k$ and $\eta_{ik} = \mathrm{diag}(1, 1, \ldots, 1)$, $i, k = 1, \ldots, D$. Here the matrices $\Lambda$ belong to the vector representation of the $O(D)$ group.

The transformations (2.1), which can be obtained continuously from the identity, form the proper Poincaré group $M_0(D, 1)$ with the elements $g = (a, \Lambda)$. Corresponding homogeneous transformations ($a = 0$) form the proper Lorentz group $SO_0(D, 1)$. In the Euclidean case, we deal with $M_0(D)$ and $SO(D)$. The composition law and the inverse element of these groups have the form

$$(a_2, \Lambda_2)(a_1, \Lambda_1) = (a_2 + \Lambda_2 a_1, \Lambda_2\Lambda_1), \quad g^{-1} = (-\Lambda^{-1}a, \Lambda^{-1}). \tag{2.2}$$

Thus, the groups $M_0(D, 1)$ and $M_0(D)$ are semidirect products

$$M_0(D, 1) = T(d) \times)\, SO_0(D, 1), \quad M_0(D) = T(D) \times)\, SO(D),$$

where $T(d)$ is the $d$-dimensional translation group.

There exists a one-to-one correspondence between the vectors $x$ and $2 \times 2$ Hermitian matrices $X$ in pseudo-Euclidean spaces of two, three, and four dimensions,[4]

$$x \leftrightarrow X, \quad X = x^\mu \sigma_\mu. \tag{2.3}$$

Namely,

$$d = 3 + 1: \quad X = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \tag{2.4}$$

$$d = 2 + 1: \quad X = \begin{pmatrix} x^0 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 \end{pmatrix}, \tag{2.5}$$

$$d = 1 + 1: \quad X = \begin{pmatrix} x^0 & x^1 \\ x^1 & x^0 \end{pmatrix}. \tag{2.6}$$

In all the above cases,

$$\det X = \eta_{\mu\nu} x^\mu x^\nu, \quad x^\mu = \frac{1}{2}\,\mathrm{Tr}(X\bar{\sigma}^\mu). \tag{2.7}$$

In Euclidean spaces of two and three dimensions, a similar correspondence has the form

$$D = 3: \quad X = \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix}, \tag{2.8}$$

$$D = 2: \quad X = \begin{pmatrix} x^2 & x^1 \\ x^1 & -x^2 \end{pmatrix}. \tag{2.9}$$

[4] We use two sets of $2 \times 2$ matrices $\sigma_\mu = (\sigma_0, \sigma_k)$ and $\bar{\sigma}_\mu = (\sigma_0, -\sigma_k)$,

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $x$ is subjected to a transformation (2.1), then $X$ transforms as follows (see, e.g., Vilenkin, 1968):

$$X' = UXU^{\dagger} + A, \tag{2.10}$$

where $A = a^{\mu}\sigma_{\mu}$ and $U$ are $2 \times 2$ complex matrices obeying the conditions

$$\sigma_{\nu}\Lambda^{\nu}_{\mu} = U\sigma_{\mu}U^{\dagger}. \tag{2.11}$$

Equation (2.11) relates the matrices $\Lambda$ and $U$. There are many $U$ that correspond to the same $\Lambda$. We may fix this arbitrariness by imposing the condition

$$\det U = 1, \tag{2.12}$$

which does not contradict the relation $\det U = e^{i\phi}$, which follows from (2.11). However, there is no one-to-one correspondence between $\Lambda$ and $U$, namely two matrices $(U, -U)$ correspond to one $\Lambda$. Considering both $U$ and $-U$ as representatives for $\Lambda$, we in fact go over from $SO_0(D, 1)$ to its double covering group Spin$(D, 1)$, or, in the Euclidean case, from $SO(D)$ to its double covering group Spin$(D)$. In the dimensions under consideration, the groups Spin$(D, 1)$ and Spin$(D)$ are isomorphic to the following ones[5]:

$$d = 3+1: \quad U \in SL(2, C), \quad U = \begin{pmatrix} u^1_1 & u^1_2 \\ u^2_1 & u^2_2 \end{pmatrix}, \qquad u^1_1 u^2_2 - u^2_1 u^1_2 = 1, \tag{2.13}$$

$$d = 2+1: \quad U \in SU(1, 1), \quad U = \begin{pmatrix} u_1 & u_2 \\ \overset{*}{u}_2 & \overset{*}{u}_1 \end{pmatrix}, \qquad |u_1|^2 - |u_2|^2 = 1, \tag{2.14}$$

$$D = 3: \quad U \in SU(2), \qquad U = \begin{pmatrix} u_1 & u_2 \\ -\overset{*}{u}_2 & \overset{*}{u}_1 \end{pmatrix}, \quad |u_1|^2 + |u_2|^2 = 1, \tag{2.15}$$

$$d = 1+1: \quad U \in SO(1, 1), \quad U = \begin{pmatrix} \cosh\left(\frac{\phi}{2}\right) & \sinh\left(\frac{\phi}{2}\right) \\ \sinh\left(\frac{\phi}{2}\right) & \cosh\left(\frac{\phi}{2}\right) \end{pmatrix}, \tag{2.16}$$

$$d = 2: \quad U \in SO(2), \qquad U = \begin{pmatrix} \cos\left(\frac{\phi}{2}\right) & \sin\left(\frac{\phi}{2}\right) \\ -\sin\left(\frac{\phi}{2}\right) & \cos\left(\frac{\phi}{2}\right) \end{pmatrix}, \tag{2.17}$$

Considering nonhomogeneous transformations and retaining both elements $U$ and $-U$, we go over from the groups $M_0(D, 1)$ and $M_0(D)$ to the groups

$$M(D, 1) = T(d) \times) \operatorname{Spin}(D, 1) \quad \text{and} \quad M(D) = T(D) \times) \operatorname{Spin}(D)$$

respectively. This allows us to avoid double-valued representations for half-integer spins. Thus, there exists a one-to-one correspondence between the elements $g$ of

[5] We denote the complex conjugation by an asterisk atop the respective quantities.

the groups $M(D, 1)$, $M(D)$, and two $2 \times 2$ matrices, $g \leftrightarrow (A, U)$. The first one, $A$, corresponds to translations and the second one, $U$, corresponds to rotations. Equation (2.10) describes the action of $M(D, 1)$ on Minkowski space [the latter is a coset space $M(D, 1)/\mathrm{Spin}(D, 1)$]. As a consequence of (2.10), we obtain the composition law and the inverse element of the groups $M(D, 1)$, $M(D)$:

$$(A_2, U_2)(A_1, U_1) = (U_2 A_1 U_2^\dagger + A_2, U_2 U_1), \quad g^{-1} = (-U^{-1} A (U^{-1})^\dagger, U^{-1}). \tag{2.18}$$

The matrices $U$ in the dimensions under consideration satisfy the following identities:

$$U \in SL(2, C): \quad \sigma_2 U \sigma_2 = (U^T)^{-1}; \tag{2.19}$$

$$U \in SU(1, 1): \quad \sigma_1 U \sigma_1 = \overset{*}{U}, \quad \sigma_2 U \sigma_2 = (U^T)^{-1}, \quad \sigma_3 U \sigma_3 = (U^\dagger)^{-1}, \tag{2.20}$$

$$U \in SU(2): \quad \sigma_2 U \sigma_2 = (U^T)^{-1} = \overset{*}{U}. \tag{2.21}$$

An equivalent picture arise in terms of the matrices $\bar{X} = x^\mu \bar{\sigma}_\mu$. Using the relation $\bar{X} = \sigma_2 X^T \sigma_2$, the transformation law for $X$, in (2.10), and the identity (2.19), one gets

$$\bar{X}' = (U^\dagger)^{-1} \bar{X} U^{-1} + \bar{A}. \tag{2.22}$$

Thus, $\bar{X}$ are transformed by means of the elements $(\bar{A}, (U^\dagger)^{-1})$. The relation $(A, U) \to (\bar{A}, (U^\dagger)^{-1})$ defines an automorphism of the Poincaré group $M(D, 1)$. In the Euclidean case, the matrices $U$ are unitary, and the latter relation is reduced to $(A, U) \to (-A, U)$.

The representation of the Poincaré transformations in the form (2.10) is closely related to a representation of finite rotations in $\mathbb{R}^d$ in terms of the Clifford algebra. In higher dimensions, the transformation law has the same form, where $A$ is a vector element and $U$ corresponds to an invertible element (spinor element) of the Clifford algebra (Benn and Tucker, 1988). The representation of the finite transformations in the form (2.10) can be useful for spin description by means of Grassmannian variables $\xi$, since $\xi$ and $\partial\xi$ give a realization of the Clifford algebra (Berezin, 1966).

### 2.2. Regular Representation and Scalar Functions on the Group

It is well known (Vilenkin, 1968; Vilenkin and Klimyk; Zhelobenko and Schtern, 1983) that any irrep of a group $G$ is contained (up to the equivalence) in a decomposition of a GRR. Thus, the study of GRR is an effective method for the analysis of irreps of the group. Consider, first, the left GRR $T_{\mathrm{L}}(g)$, which is defined in the space of functions $f(g_0)$, $g_0 \in G$, on the group as

$$T_L(g) f(g_0) = f'(g_0) = f(g^{-1} g_0), \quad g \in G. \tag{2.23}$$

As a consequence of the relation (2.23), we can write

$$f'(g_0') = f(g_0), \quad g_0' = g g_0. \tag{2.24}$$

Let $G$ be the group $M(3,1)$, and we use the parametrization of its elements by two $2 \times 2$ matrices [one Hermitian and another one from $SL(2,C)$], as described in the previous section. With such a parametrization, we use the notations

$$g \leftrightarrow (A, U), \quad g_0 \leftrightarrow (X, Z), \tag{2.25}$$

where $A$ and $X$ are $2 \times 2$ Hermitian matrices and $U, Z \in SL(2,C)$. The map $g_0 \leftrightarrow (X, Z)$ creates the correspondence

$$\begin{aligned} &g_0 \leftrightarrow (x, z, \underline{z}), \quad \text{where } x = (x^\mu), \quad z = (z_\alpha), \quad \underline{z} = (\underline{z}_\alpha), \\ &\mu = 0,1,2,3, \quad \alpha = 1,2, \quad z_1\underline{z}_2 - z_2\underline{z}_1 = 1, \end{aligned} \tag{2.26}$$

by virtue of the relations

$$X = x^\mu \sigma_\mu, \quad Z = \begin{pmatrix} z_1 & \underline{z}_1 \\ z_2 & \underline{z}_2 \end{pmatrix} \in SL(2,C). \tag{2.27}$$

On the other hand, we have the correspondence $g_0' \leftrightarrow (x', z', \underline{z}')$,

$$\begin{aligned} &g_0' = g g_0 \leftrightarrow (X', Z') = (A, U)(X, Z) = (UXU^+ + A, UZ) \leftrightarrow (x', z', \underline{z}'), \\ &x'^\mu \sigma_\mu = X' = UXU^+ + A \Longrightarrow \\ &x'^\mu = (\Lambda_0)^\mu_\nu x^\nu + a^\mu, \, \Lambda \leftarrow U \in SL(2,C), \end{aligned} \tag{2.28}$$

$$\begin{aligned} &\begin{pmatrix} z_1' & \underline{z}_1' \\ z_2' & \underline{z}_2' \end{pmatrix} = Z' = UZ \Longrightarrow \\ &z_\alpha' = U_\alpha^\beta z_\beta, \quad \underline{z}_\alpha' = U_\alpha^\beta \underline{z}_\beta, \quad U = \left(U_\alpha^\beta\right), \quad z_1'\underline{z}_2' - z_2'\underline{z}_1' = 1. \end{aligned} \tag{2.29}$$

Then the relation (2.24) takes the form

$$f'(x', z', \underline{z}') = f(x, z, \underline{z}), \tag{2.30}$$

$$x'^\mu = (\Lambda_0)^\mu_\nu x^\nu + a^\mu, \quad \Lambda \leftarrow U \in SL(2,C), \tag{2.31}$$

$$z_\alpha' = U_\alpha^\beta z_\beta, \quad \underline{z}_\alpha' = U_\alpha^\beta \underline{z}_\beta, \quad z_1\underline{z}_2 - z_2\underline{z}_1 = z_1'\underline{z}_2' - z_2'\underline{z}_1' = 1. \tag{2.32}$$

The relations (2.30)–(2.32) admit a remarkable interpretation. We may treat $x$ and $x'$ in these relations as position coordinates in Minkowski space (in different Lorentz reference frames) related by proper Poincaré transformations, and the sets

$(z, \underline{z})$ and $(z, \underline{z}')$ may be treated as spin coordinates in these Lorentz frames. They are transformed according to the formulas (2.32). Carrying the two-dimensional spinor representation of the Lorentz group, the variables $z$ and $\underline{z}$ are invariant under translations, as one can expect for spin degrees of freedom. Thus, we may treat sets $(x, z, \underline{z})$ as points in a position–spin space with the transformation law (2.31), (2.32) under the change from one Lorentz reference frame to another. In this case, Eq. (2.30)–(2.32) represent the transformation law for scalar functions on the position–spin space.

On the other hand, as we have seen, the sets $(x, z, \underline{z})$ are in one-to-one correspondence to the group $M(3, 1)$ elements. Thus, the functions $f(x, z, \underline{z})$ are still functions on this group. That is why we often call them scalar functions on the group as well, remembering that the term "scalar" came from the above interpretation.

Remember now that different functions of such type correspond to different representations of the group $M(3, 1)$. Thus, the problem of classification of all irreps of this group is reduced to the problem of a classification of all scalar functions on position–spin space. However, for the purposes of such a classification, it is natural to restrict ourselves to scalar functions that are analytic both in $z$, $\underline{z}$ and in $\overset{*}{z}$, $\overset{*}{\underline{z}}$ (or, simply speaking, that are differentiable with respect to these arguments). Such functions are denoted by $f(x, z, \underline{z}, \overset{*}{z}, \overset{*}{\underline{z}}) = f(x, \mathbf{z})$, $\mathbf{z} = (z, \underline{z}, \overset{*}{z}, \overset{*}{\underline{z}})$.

Consider the right GRR $T_{\mathrm{R}}(g)$. This representation is defined in the space of functions $f(g_0)$, $g_0 \in G$, as

$$T_{\mathrm{R}}(g) f(g_0) = f'(g_0) = f(g_0 g), \quad g \in G, \tag{2.33}$$

As a consequence of the relation (2.33), we can write

$$f'(g_0') = f(g_0), \quad g_0' = g_0 g^{-1}. \tag{2.34}$$

In the case of the proper Poincaré group, the right transformations act on $g_0 \leftrightarrow (X, Z)$ according to the formula

$$g_0' = g_0 g^{-1} \leftrightarrow (X', Z') = (X + Z^{-1} A (Z^{-1})^{\dagger}, Z U^{-1}). \tag{2.35}$$

Hence $x'^{\mu} = x^{\mu} + L^{\mu}_{\nu} a^{\nu}$, where the matrix $L$ depends on $\mathbf{z}$, $\sigma_{\nu} L^{\nu}_{\mu} = Z^{-1} \sigma_{\mu} (Z^{-1})^{\dagger}$. The transformations for $x$, $\mathbf{z}$ do not admit an interpretation similar to the left GRR case. In particular, the transformation law for $x$ does not look like a Lorentz transformation. On the other hand, the study of the right GRR is useful for the classification of the Poincaré group irreps since the generators of the right GRR are used to construct complete sets of commuting operators on the group.

## 2.3. Generators of Generalized Regular Representations

Generators of the left GRR correspond to translations and rotations. They can be written as

$$\hat{p}_\mu = -i\partial/\partial x^\mu, \quad \hat{J}_{\mu\nu} = \hat{L}_{\mu\nu} + \hat{S}_{\mu\nu}, \tag{2.36}$$

where $\hat{L}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu)$ are angular momentum operators and $\hat{S}_{\mu\nu}$ are spin operators depending on $\mathbf{z}$ and $\partial/\partial\mathbf{z}$. An explicit form of the spin operators is given in Appendix A.

The algebra of the generators (2.36) has the form

$$\begin{aligned}&[\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{J}_{\mu\nu}, \hat{p}_\rho] = i(\eta_{\nu\rho}\hat{p}_\mu - \eta_{\mu\rho}\hat{p}_\nu),\\ &[\hat{J}_{\mu\nu}, \hat{J}_{\rho\sigma}] = i\eta_{\nu\rho}\hat{J}_{\mu\sigma} - i\eta_{\mu\rho}\hat{J}_{\nu\sigma} - i\eta_{\nu\sigma}\hat{J}_{\mu\rho} + i\eta_{\mu\sigma}\hat{J}_{\nu\rho}.\end{aligned} \tag{2.37}$$

In the space of Fourier transforms

$$\varphi(p, \mathbf{z}) = (2\pi)^{-d/2}\int f(x, \mathbf{z})e^{ipx}\, dx \tag{2.38}$$

the left GRR acts as [one has to use (2.23)]

$$\begin{aligned}&T_{\mathrm{L}}(g)\varphi(p, \mathbf{z}) = e^{iap'}\varphi(p', g^{-1}\mathbf{z}), \quad p' = g^{-1}p \leftrightarrow P' = U^{-1}P(U^{-1})^\dagger,\\ &P = p_\mu\sigma^\mu.\end{aligned} \tag{2.39}$$

One can see that $\det Z$ and $\det P = p^2$ are invariant under the transformations[6] (2.39) and that $p^2$ is an eigenvalue of the Casimir operator $\hat{p}^2$.

For the groups $M(D)$ there are two types of representations, depending on $p^2$: (1) $p^2 \neq 0$, (2) $p^2 = 0$; then all $p_i = 0$, and irreps are labeled by eigenvalues of Casimir operators of the rotation subgroup.

For the groups $M(D, 1)$, there are four types of representations, depending on the eigenvalues $m^2$ of the Casimir operator $\hat{p}^2$: (1) $m^2 > 0$, (2) $m^2 < 0$ (tachyon), (3) $m^2 = 0$, $p_0 \neq 0$ (massless particle), (4) $m^2 = p_0 = 0$; irreps are labeled by eigenvalues of the Casimir operators of the Lorentz subgroup, and the corresponding functions do not depend on $x$.

For decomposing the left GRR, we construct a complete set of commuting operators in the space of functions on the group. Together with the Casimir operators, some functions of right generators[7] may be included in such a set. Therefore, it is necessary to know the explicit form of right generators. As a consequence of

[6] $p^2 = \eta^{\mu\nu}p_{\mu p_\nu}$. Since we do not use $p$ with the upper indics, this does no lead to a misunderstanding.

[7] The physical meaning of the right generators is not so transparent. However, one can remember that the right generators of $SO(3)$ in the nonrelativistic rotator theory are interpreted as operators of angular momentum in a rotating body-fixed reference frame (Biedenharm and Lauck, 1981; Landau and Lifschitz, 1977; Wigner, 1959).

the formulas

$$T_R(g) f(x, \mathbf{z}) = f(xg, \mathbf{z}g), \quad xg \leftrightarrow X + ZAZ^{\dagger}, \quad \mathbf{z}g \leftrightarrow ZU, \tag{2.40}$$

$$T_R(g)\varphi(p, \mathbf{z}) = e^{-ia'p}\varphi(p, \mathbf{z}g), \quad a' \leftrightarrow A' = ZAZ^{\dagger} \tag{2.41}$$

one obtains

$$\hat{p}^R_\mu = -(L^{-1}(\mathbf{z}))^\nu_\mu p_\nu, \quad \hat{J}^R_{\mu\nu} = \hat{S}^R_{\mu\nu}, \tag{2.42}$$

where $L \in SO(D, 1)$ [or $L \in SO(D, 1)$ in the Euclidean case]. The operators of right translations can also be written in the form $\hat{P}^R = -Z^{-1}\hat{P}(Z^{-1})^{\dagger}$; operators $\hat{S}_{\mu\nu}$ and $\hat{S}^R_{\mu\nu}$ are left and right generators of Spin$(D, 1)$ [or Spin$(D)$] and depend on $\mathbf{z}$ only. All the right generators (2.42) commute with all the left generators (2.36) and obey the same commutation relations (2.37).

In accordance with theory of harmonic analysis on Lie groups (Barut and Raczka, 1977; Zhelobenko and Schtern, 1983), there exists a complete set of commuting operators, which includes Casimir operators, a set of the left generators and a set of right generators (both sets contain the same number of generators). The total number of commuting operators is equal to the number of parameters of the group. In a decomposition of the left GRR, the nonequivalent representations are distinguished by eigenvalues of the Casimir operators, equivalent representations are distinguished by eigenvalues of the right generators, and the states inside the irrep are distinguished by eigenvalues of the left generators.

In particular, Casimir operators of the spin Lorentz subgroup are functions of $\hat{S}^R_{\mu\nu}$ (or $\hat{S}_{\mu\nu}$) and commute with all the left generators (with left translations and rotations), but do not commute with generators of the right translations. These operators distinguish equivalent representations in the decomposition of the left GRR. Aspects of the theory of harmonic analysis on the $3 + 1$ and $2 + 1$ Poincaré groups were considered in Rideau (1966), Hai (1969, 1971), and Gitman and Shelepin (1997), respectively.

If GRR acts in the space of all functions on the group $G$, a regular representation acts in the space of functions $L^2(G, \mu)$, such that the norm

$$\int \overset{*}{f}(g) f(g)\, d\mu(g) \tag{2.43}$$

is finite (Vilenkin and Klimyk, 1991; Zhelobenko and Schtern, 1983), where $d\mu(g)$ is an invariant measure on the group. The regular representation is unitary, as follows from (2.43) and from the invariance of the measure. However, we will also use nonunitary representations (in particular, finite-dimensional representations of the Lorentz group). Therefore, we consider the GRR as a more useful concept.

## 2.4. Fields on the Poincaré Group

As we have shown that the relations associated with the left GRR (2.23) define the transformation law for coordinates $(x, \mathbf{z})$ on the position–spin space under the change from one Lorentz reference frame to another. The equations

$$f'(x', \mathbf{z}') = f(x, \mathbf{z}), \tag{2.44}$$

$$x' = gx = \Lambda x + a \leftrightarrow UXU^{\dagger} + A, \quad \mathbf{z}' = g\mathbf{z} \leftrightarrow UZ. \tag{2.45}$$

define a scalar field on this space (i.e., a scalar field on the Poincaré group). In contrast to a scalar field on Minkowski space, this field is reducible with respect to both mass and spin.

We consider the transformation laws of $x$ and $\mathbf{z}$ in various dimensions in more detail.

In the two-dimensional case, matrices $Z$ depend on only one parameter [angle or hyperbolic angle, see (2.16), (2.17)]. The functions on the group depend on $x = (x^{\mu})$ and $z = e^{\alpha}$ [or $x = (x^k)$ and $z = e^{i\alpha}$ in the Euclidean case]; it is appropriate to consider these functions as functions of a real parameter $\alpha$ directly.

In the three-dimensional case, according to (2.14) and (2.15),

$$D = 3\colon \quad Z = \begin{pmatrix} z_1 & -\overset{*}{z}_{\dot{2}} \\ z_2 & \overset{*}{z}_{\dot{1}} \end{pmatrix}; \quad d = 2+1\colon \quad Z = \begin{pmatrix} z_1 & \overset{*}{z}_{\dot{2}} \\ z_2 & \overset{*}{z}_{\dot{1}} \end{pmatrix}, \quad \det Z = 1. \tag{2.46}$$

The functions $f(x, \mathbf{z})$ depend on $x = (x^{\mu})$ [in the Euclidean case $x = (x^k)$] and $\mathbf{z} = (z, \overset{*}{z})$, where $z$ are the elements of the first column of the matrix (2.46). Let us write the relation (2.45) for $d = 2+1$ in componentwise form

$$x'^{\nu}\sigma_{\nu\alpha\dot{\alpha}} = U_{\alpha}^{\beta} x^{\mu} \sigma_{\mu\beta\dot{\beta}} \overset{*}{U}{}_{\dot{\alpha}}^{\dot{\beta}} + \alpha^{\mu}\sigma_{\mu\alpha\dot{\alpha}}, \tag{2.47}$$

$$z'_{\alpha} = U_{\alpha}^{\beta} z_{\beta}, \quad \overset{*}{z}{}'_{\dot{\alpha}} = \overset{*}{U}{}_{\dot{\alpha}}^{\dot{\beta}} \overset{*}{z}_{\dot{\beta}}, \quad z'^{\alpha} = (U^{-1})_{\beta}^{\alpha} z^{\beta}, \quad \overset{*}{z}{}'^{\dot{\alpha}} = (\overset{*}{U}{}^{-1})_{\dot{\beta}}^{\dot{\alpha}} \overset{*}{z}{}^{\dot{\beta}}. \tag{2.48}$$

Undotted and dotted indices correspond respectively, to spinors transforming by means of the matrix $U$ and, the complex conjugate matrix $\overset{*}{U}$. The invariant tensor $\sigma_{\nu\alpha\dot{\alpha}}$ has one vector index and two spinor indices of distinct types.

For the group $M(3,1)$, the matrix $Z$, $\det Z = 1$, has the form (2.27); the elements $z^{\alpha}$ and $\underline{z}^{\alpha}$ of the first and second columns of the matrix (2.27) are subjected to the same transformation law. The functions $f(x, \mathbf{z})$ depend on $x = (x^{\mu})$ and $\mathbf{z} = (z, \overset{*}{z}, \underline{z}, \overset{*}{\underline{z}})$. The main reason to consider not real parameters (e.g., real and imaginary parts of $z, \underline{z}$), but $z, \underline{z}$ and $\overset{*}{z}, \overset{*}{\underline{z}}$ is the fact that the complex variables are subjected to a simple transformation rule. The use of spaces of analytic and antianalytic functions is suitable for the problem of decomposing the GRR.

According to (2.45) and (2.22), one may write the transformation law of $x^\mu, z_\alpha, \overset{*}{z}_{\dot\alpha}$ in componentwise form

$$x'^\nu \sigma_{\nu\alpha\dot\alpha} = U_\alpha^\beta x^\mu \sigma_{\mu\beta\dot\beta} \overset{*}{U}{}_{\dot\alpha}^{\dot\beta} + \alpha^\mu \sigma_{\mu\alpha\dot\alpha},$$

$$x'^\nu \bar\sigma_\nu^{\dot\alpha\alpha} = (\overset{*}{U}{}^{-1})_{\dot\beta}^{\dot\alpha} x^\mu \bar\sigma_\mu^{\dot\beta\beta} (U^{-1})_\beta^\alpha + \alpha^\mu \bar\sigma_\mu^{\dot\alpha\alpha}, \tag{2.49}$$

$$z'_\alpha = U_\alpha^\beta z_\beta, \quad \overset{*}{z}{}'_{\dot\alpha} = U_{\dot\alpha}^{\dot\beta} \overset{*}{z}_{\dot\beta}, \quad z'^\alpha = (U^{-1})_\beta^\alpha z^\beta, \quad \overset{*}{z}{}'^{\dot\alpha} = (\overset{*}{U}{}^{-1})_{\dot\beta}^{\dot\alpha} \overset{*}{z}{}^{\dot\beta}. \tag{2.50}$$

It is easy to see from (2.49) that the tensors

$$\sigma_{\mu\alpha\dot\alpha} = (\sigma_\mu)_{\alpha\dot\alpha}, \quad \bar\sigma_\mu^{\dot\alpha\alpha} = (\bar\sigma_\mu)^{\dot\alpha\alpha} \tag{2.51}$$

are invariant. These tensors are usually used to convert vector indices into spinor ones and vice versa or to construct vectors from two spinors of different types:

$$x^\mu = \frac{1}{2}\bar\sigma^{\mu\dot\alpha\alpha} x_{\dot\alpha\alpha}, \quad x_{\alpha\dot\alpha} = \sigma_{\mu\alpha\dot\alpha} x^\mu, \quad q^\mu = \frac{1}{2}\bar\sigma^{\mu\dot\alpha\alpha} z_\alpha \overset{*}{z}_{\dot\alpha}. \tag{2.52}$$

In consequence of the unimodularity of $2 \times 2$ matrices $U$, there exist invariant antisymmetric tensors $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$, $\varepsilon^{\dot\alpha\dot\beta} = -\varepsilon^{\dot\beta\dot\alpha}$, $\varepsilon^{12} = \varepsilon^{\dot1\dot2} = 1$, and $\varepsilon_{12} = \varepsilon_{\dot1\dot2} = -1$. Spinor indices are lowered and raised according to the rules

$$z_\alpha = \varepsilon_{\alpha\beta} z^\beta, \quad z^\alpha = \varepsilon^{\alpha\beta} z_\beta, \tag{2.53}$$

and in particular $\sigma_{\mu\alpha\dot\alpha} = \bar\sigma_{\mu\dot\alpha\alpha}$. Below we will also use the notations $\partial_\alpha = \partial/\partial z^\alpha$, $\partial^{\dot\alpha} = \partial/\partial \overset{*}{z}_{\dot\alpha}$ and correspondingly $\partial^\alpha = -\partial/\partial z_\alpha$, $\partial^{\dot\alpha} = -\partial/\partial \overset{*}{z}{}^{\dot\alpha}$.

In the framework of the theory of the scalar functions on the Poincaré group, *a standard spin description in terms of multicomponent functions arises under the separation of space and spin variables.*

Since $\mathbf{z}$ is invariant under translations, any function $\phi(\mathbf{z})$ carries a representation of the Lorentz group. Let a function $f(h) = f(x, \mathbf{z})$ allow the representation

$$f(x, \mathbf{z}) = \phi^n(\mathbf{z})\psi_n(x), \tag{2.54}$$

where $\phi^n(\mathbf{z})$ form a basis in the representation space of the Lorentz group. The latter means that one may decompose the functions $\phi^n(\mathbf{z}')$ of the transformed argument $\mathbf{z}' = g\mathbf{z}$ in terms of the functions $\phi^n(z)$:

$$\phi^n(\mathbf{z}') = \phi^l(\mathbf{z}) L_l^n(U). \tag{2.55}$$

An action of the Poincaré group on a line $\phi^n(\mathbf{z})\phi^n(\mathbf{z})$ is reduced to a multiplication by the matrix $L(U)$, where $U \in \mathrm{Spin}(D, 1)$, $\phi(\mathbf{z}') = \phi(\mathbf{z})L(U)$.

Comparing the decompositions of the function $f'(x', \mathbf{z}') = f(x, \mathbf{z})$ over the transformed basis $\phi(\mathbf{z}')$ and over the initial basis $\phi(\mathbf{z})$,

$$f'(x', \mathbf{z}') = \phi(\mathbf{z}')\psi'(x') = \phi(\mathbf{z})L(U)\psi'(x') = \phi(\mathbf{z})\psi(x),$$

where $\psi(x)$ is a column with components $\psi_n(x)$, one obtains

$$\psi'(x') = L(U^{-1})\psi(x), \tag{2.56}$$

that is, the transformation law of a tensor field on Minkowski space. This law corresponds to the representation of the Poincaré group acting in a linear space of tensor fields as follows: $T(g)\psi(x) = L(U^{-1})\psi(\Lambda^{-1}(x - a))$. According to (2.55) and (2.56), the functions $\phi(z)$ and $\psi(x)$ transform under contragradient representations of the Lorentz group.

For example, let us consider scalar functions on the Poincaré group $f_1(x, \mathbf{z}) = \psi_\alpha(x)z^\alpha$ and $f_2(x, \mathbf{z}) = \bar{\psi}_\alpha(x)\overset{*}{z}^\alpha$, which correspond to spinor representations of Lorentz group. According to (2.54) and (2.56),

$$\psi'_\alpha(x') = U_\alpha^\beta \psi_\beta(x), \quad \bar{\psi}'_{\dot\alpha}(x') = \overset{*}{U}{}_{\dot\alpha}^{\dot\beta}\, \bar{\psi}_{\dot\beta}(x). \tag{2.57}$$

The product $\psi_\alpha(x)\bar{\psi}^{*\alpha}(x)$ is Poincaré invariant.

Thus, tensor fields of all spins are contained in the decomposition of the field (2.44) on the Poincaré group, and the problems of their classification and of the construction of explicit realizations are reduced to the problem of the decomposition of the left GRR.

Notice that above we reject the phase transformations, which correspond to $U = e^{i\phi}$. These transformations of the $U(1)$ group do not change space-time coordinates $x$, but change the phase of $\mathbf{z}$. According to (2.55) and (2.56), this leads to the transformation of the phase of the tensor field components $\psi_n(x)$. Taking account of this transformation means considering the functions on the group $T(d) \times)$ Spin$(D, 1) \times U(1)$.

## 2.5. Automorphisms of the Poincaré Group and Discrete Transformations: *P, C, T*

Let us consider elements $g \leftrightarrow (A, U)$, $g_0 \leftrightarrow (X, Z)$ of the Poincaré group $M(D, 1)$. It is easy to see that the transformations

$$(A, U) \to (\bar{A}, (U^\dagger)^{-1}), \quad (X, Z) \to (\bar{X}, (Z^\dagger)^{-1}), \tag{2.58}$$

$$(A, U) \to (\overset{*}{A}, \overset{*}{U}), \quad (X, Z) \to (\overset{*}{X}, \overset{*}{Z}), \tag{2.59}$$

$$(A, U) \to (-A, U), \quad (X, Z) \to (-X, Z) \tag{2.60}$$

are outer involutory automorphisms of the group and generate a finite group consisting of eight elements.

The automorphisms (2.58)–(2.60) define discrete transformations of space-time and spin coordinates $x, \mathbf{z}$. The substitution of transformed coordinates into the functions $f(x, \mathbf{z})$ [or into the generators (2.36)] leads to a change in sign of some physical variables. (Notice that the substitution both into the functions and into the generators leaves signs unaltered.)

The space reflection (or parity transformation $P$) is defined by the relations $x^0 \to x^0$, $x^k \to -x^k$, or $X \to \bar{X}$. If $X$ is transformed by means of the group element $(A, U)$, then $\bar{X}$ is transformed by means of the group element $(\bar{A}, (U^\dagger)^{-1})$; see (2.22). Therefore, the space reflection represents a realization of the automorphism (2.58) of the Poincaré group

$$(X, Z) \xrightarrow{P} (\bar{X}, (Z^\dagger)^{-1}). \tag{2.61}$$

Thus, under the space reflection, $x$ and $\mathbf{z}$ have to be changed in all the constructions according to (2.61). In particular, for the momentum $P = p_\mu \sigma^\mu$, we obtain $P \to \bar{P}$, where $\bar{P} = p_\mu \bar{\sigma}^\mu$. The generators of the rotations are not changed, and the generators of the boosts change their signs only.

The time reflection transformation $T'$ is defined by the relation $x^\mu \to (-1)^{\sigma_{0\mu}} x^\mu$, or $X \to -\bar{X}$, and corresponds to the composition of automorphims (2.58) and (2.60):

$$(X, Z) \xrightarrow{T'} (-\bar{X}, (Z^\dagger)^{-1}). \tag{2.62}$$

Inversion $PT'$, $(X, Z) \xrightarrow{PT'} (-X, Z)$, corresponds to the automorphism (2.60).

Automorphism of the complex conjugation (2.59) means the substitution $i \to -i$,

$$f(x, \mathbf{z}) \xrightarrow{C} \overset{*}{f}(x, \mathbf{z}). \tag{2.63}$$

One can show that in the framework of the characteristics related to the Poincaré group, this transformation corresponds to the charge conjugation. Both the transformation (2.63) and the charge conjugation change the signs of all the generators, $\hat{p}_\mu \to -\hat{p}_\mu$, $\hat{L}_{\mu\nu} \to -\hat{L}_{\mu\nu}$, $\hat{S}_{\mu\nu} \to -\hat{S}_{\mu\nu}$. Below, considering RWE, we will see that the transformation (2.63) also changes the sign of the current vector $j^\mu$.

The time reversal $T$ is defined by the relation $X \to -\bar{X}$ (the time reflection transformation $T'$), with the supplementary condition of energy sign conservation, which means $P \to \bar{P}$. Therefore, we have the conditions $\hat{p}_\mu \to -(-1)^{\delta_{0\mu}} \hat{p}_\mu$, $\hat{L}_{\mu\nu} \to -(-1)^{\delta_{0\mu}+\delta_{0\nu}} \hat{L}_{\mu\nu}$, and $\hat{S}_{\mu\nu} \to -(-1)^{\delta_{0\mu}+\delta_{0\nu}} \hat{S}_{\mu\nu}$. The transformation $CT'$ obeys these conditions.

However, it is known (Kemmer *et al.*, 1959; Umezava *et al.*, 1954) that it is possible to give two distinct definitions of the time-reversal transformation obeying the above conditions. Wigner time reversal $T_W$ leaves the total charge (and correspondingly $j^0$) unaltered, and reverses the direction of current $j^k$. Schwinger time reversal $T_{Sch}$ (Schwinger, 1951) leaves the current $j^k$ invariant and reverses the charge.

The transformation $CT'$ changes the sign of $j^0$ and therefore can be identified with Schwinger time reversal, $T_{Sch} = CT'$. The $CPT_{Sch}$ transformation corresponds to the inversion $(X, Z) \to (-X, Z)$. The Wigner time reversal $T_W$ and $CPT_W$ transformation can be defined considering both outer and inner automorphisms of the proper Poincaré group (Buchbinder *et al.*, 2000b). Namely, $CPT_W = I_x I_z$, where $I_z$ is defined as

$$(X, Z) \stackrel{I_z}{\to} (X, Z(-i\sigma_2)) \tag{2.64}$$

and is a composition of the inner automorphism $(X, Z) \to (\overline{X}^T, (Z^T)^{-1})$ and of the rotation by the angle $\pi$. Wigner time reversal is the composition of the above transformations, $T_W = I_z CT = I_z T_{Sch}$.

The improper Poincaré group is defined as a group that includes continuous transformations of the proper Poincaré group $g \in M(D, 1)$ and the space reflection $P$.

In the Euclidean case, the space reflection is reduced to the substitution $(X, Z) \stackrel{P}{\to} (-X, Z)$. The charge conjugation inverts the momentum and spin orientation.

## 2.6. Equivalent Representations

In the decomposition of the scalar field (2.44) on the Poincaré group (or, which is the same, of the left GRR), there are equivalent representations distinguished by the right generators.

Remember that representations $T_1(g)$ and $T_2(g)$ acting in linear spaces $L_1$ and $L_2$, respectively, are equivalent if there exists an invertible linear operator $A$: $L_1 \to L_2$ such that

$$AT_1(g) = T_2(g)A. \tag{2.65}$$

In particular, the left and right GRR of a Lee group $G$ are equivalent. The operator $(Af)(g) = f(g^{-1})$ realizes the equivalence (Vilenkin, 1968; Zhelobenko and Schtern, 1983).

Let us consider functions $f(x, \mathbf{z})$ belonging to two equivalent representations in the decomposition of the left GRR of the group $M(D, 1)$ [or $M(D)$]. If the representations $T_1(g)$ and $T_2(g)$ acting in the different subspaces $L_1$ and $L_2$ of the space of functions on the group are equivalent, then

$$AT_1(g) f_1(x, \mathbf{z}) = T_2(g)Af_1(x, \mathbf{z}), \qquad f_2(x, \mathbf{z}) = Af_1(x, \mathbf{z}),$$

where $f_1(x, \mathbf{z}) \in L_1$ and $f_2(x, \mathbf{z}) \in L_2$. In particular, if the operator $A: L_1 \to L_2$ is a function of the right translation generators $\hat{p}^{\mathrm{R}}_{\mu}$, then one cannot map the function $f_1(x, \mathbf{z})$ to the function $f_2(x, \mathbf{z})$ by the group transformation, which leaves the interval square invariant. Therefore, the physical equivalence of the states, which corresponds to equivalent irreps in the decomposition of the scalar field $f(x, \mathbf{z})$, is not evident.

Below we will consider a number of examples in various dimensions. In particular, in the framework of the representation theory of the three-dimensional Euclidean group $M(3)$, irreps characterized by different spins (but with the same spin projection on the direction of propagation) are equivalent. There are no contradictions in the fact that in this case, different particles are described by equivalent irreps since it is not possible to map corresponding wave functions into one another by the rotations or translations of the frame of reference.

In some cases, more general considerations may be based on the representation theory of an extended group. In the framework of the latter, there are two possibilities: either irreps labeled by different eigenvalues of right generators of the initial group are nonequivalent, or some equivalent irreps of the initial group are combined into one irrep. For example, in nonrelativistic theory, spin becomes the characteristic of nonequivalent irreps after the extension of $M(3)$ up to the Galilei group. In $3+1$ dimensions, for $m > 0$, the proper Poincaré group representations characterized by different chiralities are equivalent. If we go from the Lorentz group to the group $SO(3, 2)$, then all states characterized by spin $s$ with different chiralities $\lambda$, $\lambda = -s, -s+1, \ldots, s$, are combined into one irrep.

The space of functions $f(x, \mathbf{z})$ contains functions transforming under equivalent representations of the proper Poincaré group and is sufficiently wide to define discrete transformations, including space reflection, time reflection, and charge conjugation. These discrete transformations associated with automorphisms of the group also combine equivalent irreps of the proper Poincaré group into one representation of the extended group. For example, in $3+1$ dimensions, space reflection combines two equivalent irreps of the proper group labeled by $\lambda$ and $-\lambda$ into one irrep of the improper group.

As we will see below, the different types of RWE (finite-component and infinite-component equations) are also associated with equivalent representations in the decomposition of the left GRR.

Thus, initially, it is appropriate to consider all representations in the decomposition of the scalar field on the Poincaré group, including equivalent ones. In this sense we note the close analogy with the theory of the nonrelativistic three-dimensional rotator (Biedenharn and Louck, 1981; Landau and Lifschitz, 1977; Wigner, 1959). In the latter theory, one considers functions on the rotation group $SU(2)$ and two sets of operators: angular momentum operators in an inertial laboratory (space-fixed) frame (left generators $\hat{J}^{\mathrm{L}}_i$) and angular momentum operators in a rotating (body-fixed) frame (right generators $\hat{J}^{\mathrm{R}}_i$). The classification of the rotator states is based on the use of the complete set of commuting operators, which, besides $\hat{\mathbf{J}}^2$ and $\hat{J}_3^{\mathrm{L}}$, includes also $\hat{J}_3^{\mathrm{R}}$. The operator $\hat{J}_3^{\mathrm{R}}$ distinguishes equivalent representations in the decomposition of the left GGR of the rotation group and corresponds to a quantum number that does not depend on the choice of the laboratory frame. This quantum number plays a significant role in the theory of molecular spectra. In the $3+1$-dimensional case, there exist two analogs of $\hat{J}_3^{\mathrm{R}}$, namely $\hat{B}_3^{\mathrm{R}} = \hat{S}_{03}^{\mathrm{R}}$ and $\hat{S}_3^{\mathrm{R}} = \hat{S}_{12}^{\mathrm{R}}$, which act in the space of functions on the Poincaré group. As we will see below, the first may be interpreted as a chirality operator, and the second allows us to distinguish particles and antiparticles.

## 2.7. Quasiregular Representations and Spin Description

The consideration of GRR of the Poincaré group ensures the possibility of a consistent description of particles with arbitrary spin by means of scalar functions on $\mathbb{R}^d \times \mathrm{Spin}(D, 1)$. At the same time, for the description of spinning particles, it is possible to use the spaces $\mathbb{R}^d \times M$, where $M$ is a homogeneous space of the Lorentz group (one- or two-sheeted hyperboloid, cone, complex disk, projective space, and so on); see, for example, Bácry and Kihlberg (1969), Kihlberg (1970), Boyer and Fleming (1974), Wigner (1963), Kim and Wigner (1987), Biedenharn *et al.* (1988), Haslewicz and Siemion (1992), Kuzenko *et al.* (1995), Lyakhovich *et al.* (1996), Deriglazov and Gitman (1999), Drechsler (1997), and Jackiw and Nair (1991), Plyushchay (1991, 1992), Cortes and Plyushchay (1996) for the $3+1$- and $2+1$-demensional cases, respectively. In some work, fields on homogeneous spaces are considered; in other work, such spaces are treated as phase spaces of a classical mechanics, and the latter are treated as models of spinning relativistic particles.

These spaces appear in the framework of the next group-theoretic scheme. Let us consider the left quasiregular representation of the Poincaré group

$$T(g) f(g_0 H) = f(g^{-1} g_0 H), \quad H \subset \mathrm{Spin}(D, 1). \tag{2.66}$$

$H$ is a subgroup of $\mathrm{Spin}(D, 1)$, and since $x$ is invariant under right rotations [see (2.40)],

$$g_0 \leftrightarrow (X, Z), \quad g_0 H \leftrightarrow (X, ZH).$$

Therefore, the relation (2.66) defines the representation of the Poincaré group in the space of functions $f(x, zH)$ on

$$\mathbb{R}^d \times (\mathrm{Spin}(D, 1)/H). \tag{2.67}$$

In the decomposition of the representation in the space of functions on $\mathrm{Spin}(D, 1)/H$ [or $\mathbb{R}^d \times (\mathrm{Spin}(D, 1)/H)$], there is, generally speaking, only part of the irreps of the Lorentz (or Poincaré) group. In particular, the case $H \sim \mathrm{Spin}(D, 1)$ corresponds to a scalar field on Minkowski space. The classification and description of homogeneous spaces of $3+1$ Poincaré and Lorentz groups can be found in Finkelstein (1955), Bacry and Kihlberg (1969), and Gel'fand *et al.* (1966).

Thus, the consideration of quasiregular representations allows one to construct a number of spin models classified by subgroups $H \subset \text{Spin}(D, 1)$. But the existence of a nontrivial subgroup $H$ leads to the rejection of part of the equivalent (with different characteristics with respect to the Lorentz subgroup) or, possibly, nonequivalent irreps of the Poincaré group.

## 2.8. Relativistic Wave Equations

The problem of RWE construction for particles with arbitrary spin in various dimensions is far from resolved and continues to attract significant attention. To describe massive particles of spin $s$ in four dimensions, one usually employs the equations connected with the representations $(\frac{s}{2}\ \frac{s}{2})$ and $(\frac{2s\pm1}{4}\ \frac{2s\mp1}{4})$ of the Lorentz group (see, e.g., Ohnuki, 1988; Buchbinder and Kuzenko, 1995). These equations admit Lagrangian formulations (Fierz and Pauli, 1939; Singh and Hagen, 1974a, b), but for $s > 1$, minimal electromagnetic coupling leads to noncasual propagation (Wightman, 1978; Zwanziger, 1978). On the other hand, all known equations with casual solutions either have a redundant number of independent components [as the equations (Hurley, 1971; Kruglov, 1999) for representations $(s\,0)$ and $(0\,s)$ have] or describe many masses and spins simultaneously, as the Bhabha equations (Lubanski, 1942; Bhabha, 1945; Krajcik and Nieto, 1977) do. Besides the problem of interaction of higher spin fields, one may mention attempts to construct RWE with a completely positive energy spectrum (Majorana, 1932; Gel'fand *et al.*, 1963; Stoyanov and Todorov, 1968; Dirac, 1971, 1972a) and RWE for fractional spin fields (Jackiw and Nair, 1991; Plyushchay, 1991, 1992; Gitman and Shelepin, 1997).

With respect to the mathematical methods used, it is possible to divide approaches to RWE construction into three groups.

The first approach, which follows Dirac (1936), Fierz and Pauli (1939), Rarita and Schwinger (1941), and Bargmann and Wigner (1948), deals with equations for symmetric spin tensors. It allows one to describe fields with fixed mass and spin and also to construct RWE that admit Lagrangian formulation; however, as mentioned above, for $s > 1$, we face the problem of noncasual propagation.

The second approach, which follows Kemmer (1939), Lubanski (1942), Bhabha (1945), Harish-Chandra (1947), Gel'fand and Yaglom (1948), and Gel'fand *et al.* (1963) is devolted to studying RWE of the form $(\alpha^\mu \hat{p}_\mu - \varkappa)\psi(x) = 0$, and is based on the use of algebraic properties of $\alpha$-matrices. These equations admit Lagrangian formulation; however, for $s > 1$, they describe a nonphysical spectrum of particles: a decreasing mass with increasing spin.

The third approach is connected to the use of supplementary variables to describe spin degrees of freedom and initially was suggested for RWE with a mass spectrum (Ginzburg and Tamm, 1947; Ginzburg, 1956). It was used for constructing positive-energy wave equations (Dirac, 1971, 1972a; Stoyanov and Todorov, 1968), equations describing gauge fields (Vasiliev, 1992, 1996), and anyon equations (Gitman and Shelepin, 1997; Jackiw and Nair, 1991; Plyushchay, 1991, 1992).

From the point of view of the approach that we developed above, the problem of constructing RWE looks like the selection of invariant subspaces in the space of functions on the group.

The classification of the scalar functions can be based on the use of the operators $\hat{C}_k$ commuting with $T_{\rm L}(g)$ (and correspondingly with all the left generators). For these operators, as a consequence of the relation $\hat{C}f(x, \mathbf{z}) = cf(x, \mathbf{z})$, one obtains that $\hat{C} f'(x, \mathbf{z}) = cf'(x, \mathbf{z})$, where $f'(x, \mathbf{z}) = T_{\rm L}(g) f(x, \mathbf{z})$. Therefore, different eigenvalues $c$ correspond to subspaces that are invariant with respect to the action of $T_{\rm L}(g)$. The invariant subspaces correspond to subrepresentations of the left GRR.

In addition to the Casimir operators, for the classification, one may use the right generators since all the right generators commute with all the left generators. The right generators, as mentioned, distinguish equivalent representations in the decomposition of the left GRR.

There is some freedom to choose commuting operators that are functions of the right generators of the Poincaré group. We will use only functions of the generators of the right rotations (2.42), in particular, for coordination with the standard formulation of the theory.

Following the general scheme of harmonic analysis, for $M(D, 1)$, one may consider the system consisting of $d$ equations

$$\hat{C}_k f(x, \mathbf{z}) = c_k f(x, \mathbf{z}), \qquad (2.68)$$

where $\hat{C}_k$ are the Casimir operators of the Poincaré group and of the spin Lorentz subgroup. These operators constitute a subset of the complete set of commuting operators on the Poincaré group. This is the system we will use for $d = 2 + 1$ below.

On the other hand, there exist additional requirements associated with the physical interpretation. In the first place, in the massive case, the system must be invariant under space reflection in order to describe states with definite parity. Second, it is often supposed that the system contains an equation of first order in $\partial/\partial t$ [the approach based on the first-order equations advocated mainly in Bhabha (1945), Krajcik and Nieto (1976), and Biritz (1979).[8]

[8] As a consequence of relativistic invariance, an equation linear in $\partial/\partial t$ can be either first order or infinite order in space derivatives [square-root Klein–Gordon equation (Briegel *et al.*, 1991; Samarov, 1984; Smith, 1993; Sucher, 1963)]. The latter type of equation is naturally obtained in the theory of Markov processes for probability amplitudes (Shelepin, 1997).

The Casimir operators of the Poincaré group are functions of the generators $\hat{p}_\mu$ and $\hat{J}_{\mu\nu}$. In odd dimensions, there exists a Casimir operator linear in $\hat{p}_\mu$ since the invariant tensor $\varepsilon^{\mu\cdots\nu}$ also has an odd number of indices. As we will see below, in $2+1$ dimensions, the system (2.68) is invariant under space reflections.

In even dimensions, the invariant tensor $\varepsilon^{\mu\cdots\nu}$ also has an even number of indices, and therefore a Casimir operator linear in $\hat{p}_\mu$ does not exist. In even dimensions under space reflection, the irrep of the proper Poincaré group is mapped onto an equivalent irrep labeled by other eigenvalues of the Casimir operators of the spin Lorentz subgroup. The linear combinations of basis elements of these two irreps form the bases of two $M = \pm 1$ irreps labeled by intrinsic parity of the improper Poincaré group including space reflection.

In even dimensions, there exists an operator $\hat{C}' = \hat{p}_\mu \hat{\Gamma}^\mu$, where $\hat{\Gamma}^\mu = \hat{\Gamma}^\mu(\mathbf{z}, \partial/\partial\mathbf{z})$, commuting with all left generators and connecting the states that are interchanged under space reflections. In contrast to the Casimir operators, this operator is not a function of generators of the Poincaré group and does not commute with some right generators. A first-order equation

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x, \mathbf{z}) = \varkappa f(x, \mathbf{z}) \tag{2.69}$$

connects two irreps of the group $M(D, 1)$ characterized by different eigenvalues of the Casimir operator of the spin Lorentz subgroup. Equations (2.68) and (2.69) have the same form; namely, the invariant operator acts on the scalar function $f(x, \mathbf{z})$ on the group $M(D, 1)$. The addition of the operators $\hat{\Gamma}^\mu$ means in fact the extension of the Lorentz group to a wider group [in particular, in four dimensions, to the $3+2$ de Sitter group $SO(3, 2)$]. Equation (2.69) replaces equations of the system (2.68), which are not invariant under space reflection.

In the approach under consideration, equations have the same form for all spins. The separation of the components with fixed spin and mass is realized by fixing eigenvalues of the Casimir operators of the Poincaré group (or the operator $\hat{p}_\mu \hat{\Gamma}^\mu$). Fixing the representation of the Lorentz group under which $\phi(\mathbf{z})$ transforms in the decomposition

$$f(x, \mathbf{z}) = \phi^n(\mathbf{z})\psi_n(x),$$

one obtains RWE in standard multicomponent form. This fixing is realized by the Casimir operator of the spin Lorentz subgroup.

There are two types of equations to describe one and the same spin, one on functions $f(x, \mathbf{z})$, where $\phi^n(\mathbf{z})$ transforms under the finite-dimensional nonunitary irrep of the Lorentz group, and another on functions $f(x, \mathbf{z})$, where $\phi^n(\mathbf{z})$ transforms under the infinite-dimensional unitary irrep of the Lorentz group. In the matrix representation, these equations are written in the form of finite-component or infinite-component equations, respectively. The latter type of equation [e.g., the Majorana equations (Fradkin, 1966; Gel'fand *et al.*, 1963; Majorana, 1932; Stoyanov and Todorov, 1968)] is interesting because it gives the possibility to combine relativistic invariance with a probability interpretation. The desirability of this combination was emphasized in Dirac (1972b).

Let us briefly consider the possibility of the existence of particles with fractional spin. The restrictions on the spin value arise in the representation theory of $M(D)$ and $M(D, 1)$ if one restricts the consideration to (1) unitary, (2) finite-dimensional (with respect to the number of spin components), or (3) single-valued representations. (The latter means that the representation acts in the space of single-valued functions). The restriction to single-valued functions (often supposed in mathematical papers related to a classification of representations) is omitted in some physical problems, which allows one to consider particles with fractional spin (anyons). Thus, we will also consider multivalued representations of $M(D)$ and $M(D, 1)$ in the space of the functions $f(x, \mathbf{z})$ on the group. These representations correspond to single-valued representations of the universal convering group.

## 3. TWO-DIMENSIONAL CASE

### 3.1. Field on the Group *M*(2)

In the two-dimensional case, the general formulas become simpler. The matrices $U$ (2.17) of the $SO(2)$ subgroup depend on only one parameter, namely the angle $\phi$, $0 \le \phi \le 4\pi$. Using the correspondence $g_0 \leftrightarrow (X, Y(\theta/2))$, $g \leftrightarrow (A, U(\phi/2))$, one may write the action of GRR:

$$T_{\mathrm{L}}(g) f(x, \theta/2) = f(x', \theta/2 - \phi/2), \tag{3.1}$$

$$x_1' = (x_1 - a_1)\cos\phi + (x_2 - a_2)\sin\phi, \quad x_2' = (x_2 - a_2)$$
$$\cos\phi - (x_1 - a_1)\sin\phi, \quad T_{\mathrm{R}}(g) f(x, \theta/2) = f(x'', \theta/2 + \phi/2), \tag{3.2}$$

$$x_1'' = x_1 + a_1\cos\theta - a_2\sin\theta, \quad x_2'' = x_2 + a_2\cos\theta + a_1\sin\theta.$$

Left and right generators that correspond to the parameters $\theta$ and $\phi$ are given by

$$\hat{p}_k = -i\partial_k, \quad \hat{J} = \hat{L} + \hat{S}, \tag{3.3}$$

$$\hat{p}_k^{\mathrm{R}} = i\Lambda_k^i\partial_i, \quad \hat{J}^{\mathrm{R}} = -\hat{S}, \tag{3.4}$$

where

$$\hat{L} = i(x_1\partial_2 - x_2\partial_1) = -i\frac{\partial}{\partial\varphi}, \quad \hat{S} = -i\frac{\partial}{\partial\theta}, \quad \Lambda = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

The functions on the group are those on $\mathbb{R}^2 \times S^1$, and the invariant measure on the group is

$$d\mu(x, \theta) = (4\pi)^{-1}\, dx_1\, dx_2\, d\theta, \quad -\infty < x < +\infty \quad 0 \le \theta < 4\pi.$$

We will consider two complete sets of commuting operators: $\hat{p}_1$, $\hat{p}_2$, $\hat{S}$ and $\hat{p}^2$, $\hat{J}$, $\hat{S}$. The eigenfunctions of these operators are

$$\langle x_1 x_2 \theta \mid p_1 p_2 s\rangle = (2\pi)^{-1}\exp(ip_1 x_1 + ip_2 x_2 + is\theta), \tag{3.5}$$

$$\langle r\varphi\theta \mid pjs\rangle = (2\pi)^{-1/2} i^l J_l(pr)\exp(il\varphi)\exp(is\theta), \tag{3.6}$$

where $l = j - s$ is the orbital momentum and $J_l(pr)$ is the Bessel function. Irreps are labeled by eigenvalues $p^2$ of the Casimir operator $\hat{p}^2$. For $p \neq 0$, the representation is irreducible; for $p = 0$, it decomposes into one-dimensional irreps of spin subgroup $U(1)$, which are labeled by eigenvalues $s$ of the spin projection operator (or, simply speaking, the spin operator) $\hat{S}$.

For $p \neq 0$, the representations characterized by the spins $s$ and $s' = s + n$, where $n$ is an integer, are equivalent. The operator $\hat{S}$ commutes with all left generators, but does not commute with the generators of right translations, which mix spin and space coordinates. Operators $\hat{p}_+^{\mathrm{R}} = p_1^{\mathrm{R}} - ip_2^{\mathrm{R}}$ and $\hat{p}_-^{\mathrm{R}} = p_1^{\mathrm{R}} + ip_2^{\mathrm{R}}$ are raising and lowering operators with respect to spin $s$,

$$\hat{p}_\pm^{\mathrm{R}} \mid p_1 p_2 s\rangle = (ip_1 \pm p_2) \mid p_1 p_2 s \pm 1\rangle. \tag{3.7}$$

Right translations do not conserve both interval (distance) and spin $s$.

The functions (3.6) satisfy the relations of orthogonality and completeness

$$\int \langle pjs \mid r\varphi\theta\rangle\langle r\varphi\theta \mid pjs\rangle r\, dr\, d\varphi\, d\theta = \frac{\delta(p - p')}{p}\delta_{jj'}\delta_{ss'}, \tag{3.8}$$

$$\int \sum_{l,s}\langle r\varphi\theta \mid pjs\rangle\langle pjs \mid r\varphi\theta\rangle\, dp = \frac{\delta(r - r')}{r}\delta(\varphi - \varphi')\delta(\theta - \theta'). \tag{3.9}$$

This means that we have obtained the decomposition of the left regular representation. The spin operator $\hat{S}$ distinguishes equivalent irreps (except for the case $p = 0$, when irreps are labeled by its eigenvalues). The decomposition of the functions of $\theta$ on the eigenfunctions of $\hat{S}$ corresponds to the Fourier series expansion of functions on a circle.

Thus, the representations of $M(2)$ are single-valued for integer and half-integer $s$. Fractional values of $s$ correspond to multivalued representations. Irreps are equivalent if they are labeled by the same $p \neq 0$ and the difference $s - s' = n$ is an integer. For fixed $p \neq 0$, there are only two nonequivalent single-valued representations, which correspond to integer and half-integer spin. Nonequivalent multivalued representations for fixed $p \neq 0$ are labeled by $\tilde{s} \in [0, 1)$, $\tilde{s} = s - [s]$.

### 3.2. Field on the Group $M(1, 1)$

Matrices $U$ (2.16) of the $SO(1, 1)$ subgroup, which is isomorphic to an additive group of real numbers, depend on the hyperbolic angle $\phi$. Using the correspondence

$g_0 \leftrightarrow (X, Z(\theta/2))$, $g \leftrightarrow (A, U(\phi/2))$, we write the action of GRR:

$$T_{\rm L}(g) f(x, \theta/2) = f(x', \theta/2 - \phi/2), \tag{3.10}$$

$$x'^0 = (x^0 - a^0)\cosh\phi + (x^1 - a^1)\sinh\phi,$$
$$x'^1 = (x^1 - a^1)\cosh\phi + (x^0 - a^0)\sinh\phi,$$

$$T_{\rm R}(g) f(x, \theta/2) = f(x'', \theta/2 + \phi/2), \tag{3.11}$$

$$x''^0 = x^0 + a^0\cosh\theta - a^1\sinh\theta,$$
$$x''^1 = x^1 + a^1\cosh\theta - a^0\sinh\theta.$$

The functions on the group are the those on $\mathbb{R}^2 \times \mathbb{R}$, and the invariant measure on the group can be written as

$$d\mu(x, \theta) = dx^0\, dx^1\, d\theta, \quad -\infty < x, \theta < +\infty.$$

As above, we will consider two complete sets of commuting operators, $\hat{p}_1$, $\hat{p}_2$, $\hat{S}$ and $\hat{p}^2, \hat{J}, \hat{S}$, where $\hat{J} = \hat{L} + \hat{S}$, $\hat{L} = i(x^0\partial^0 + x^1\partial^1)$, $\hat{S} = -i\partial/\partial\theta$. The eigenfunctions of the first set are

$$\langle x^0 x^1 \theta \mid p_1 p_2 \lambda\rangle = (2\pi)^{-3/2}\exp(ip_\mu x^\mu + i\lambda\theta), \tag{3.12}$$

where $\lambda$ is an eigenvalue of the spin projection (chirality) operator $\hat{S}$. The form of the eigenfunctions of the second set depends on the type of irrep. There are four types of unitary irreps labeled by the eigenvalue $m^2$ of the operator $\hat{p}^2$ [80].

1. $m^2 > 0$. Representations correspond to particles of nonzero mass; the eigenfunctions of the operators $\hat{p}^2, \hat{J}, \hat{S}$ are

$$\langle r\varphi\theta \mid mj\lambda\rangle = (4\pi)^{-1} i \exp(\pi l/2) H^{(2)}_{il}(\pm mr)\exp(il\varphi)\exp(i\lambda\theta), \tag{3.13}$$

   where $H^{(2)}_{il}(mr)$ is the Hankel function, $r^2 = (x^0)^2 - (x^1)^2$, and $\pm$ corresponds to the sign of energy $p_0$.
2. $m^2 < 0$. Representations correspond to tachyons, which in $d = 1 + 1$, are more similar to usual particles because of the symmetry between space and time variables. The form of $\langle r\varphi\theta \mid mj\lambda\rangle$ coincides with (3.13), but $m$ is imaginary.
3. $m = 0$, $p_1 = \pm p_0$. Representations correspond to massless particles. According to (2.39), for the action of finite transformations $T_0(g)$ on the functions $f(p, \pm p, \theta/2)$, one obtains

$$T_0(g) f(p, \pm p, \theta/2) = e^{iap'} f(p', \pm p', \theta/2 - \phi/2), \quad p' = e^{\mp\phi} p.$$

   Therefore, the representation $T_0(g)$ is reducible and splits into four irreps differing in the signs of $p_0$ and $p_1 = \pm p_0$, and an reducible representation that corresponds to $m = p_0 = 0$.

4. $m = p_0 = 0$. This representation decomposes into a sum of one-dimensional irreps of the group $SO(1,1)$, which are labeled by eigenvalues of $\hat{S}$.

There are no integer value restrictions for the spectrum of $\hat{S}$, and chirality can be fractional, $-\infty < \lambda < +\infty$. The decomposition of the functions $f(x,\theta)$ in terms of the eigenfunctions of $\hat{S}$ corresponds to the Fourier integral expansion of functions on a line. The equivalence of the representations characterized by different $\lambda$ is related to the fact that, as in the Euclidean case, the operator $\hat{S}$ does not commute with right translations.

One can convert vector indices into spinor indices and vice versa with the help of the formula (2.10). In the case under consideration, matrices $U$ are real and symmetric, $X' = UXU$, or in componentwise form, $x'^{\nu}\sigma_{\nu\alpha\alpha'} = U_{\alpha}^{\beta}\sigma_{\mu\beta\beta'}x^{\mu}U_{\alpha'}^{\beta'}$, and there exists one type of spinor index only. Denoting elements of the first column of the matrix $Z$ transforming under the spinor representation of $SO(1,1)$ by $z_\alpha$, $z_1 = \cosh(\theta/2)$, $z_2 = \sinh(\theta/2)$, we obtain for the components of the vector and antisymmetric tensor

$$q^{\mu} = \sigma^{\mu\alpha\beta}z_\alpha z_\beta, \quad q^0 = \cosh\theta, \quad q^1 = \sinh\theta, \quad q^{01} = \sigma^{01\alpha\beta}z_\alpha z_\beta = i. \tag{3.14}$$

There exist two invariant tensors $\eta^{\mu\nu}$ and $\varepsilon^{\mu\nu}$, which can be used for raising of indices. This is related to the fact that the vectors $(x^0\ x^1)$ and $(x^1\ x^0)$ have the same transformation rule, and one can construct an invariant from two vectors in two different ways: $\eta^{\mu\nu}q_\mu q'_\nu = \cosh(\theta-\theta')$, $\varepsilon^{\mu\nu}q_\mu q'_\nu = \sinh(\theta-\theta')$.

### 3.3. Relativistic Wave Equations in 1 + 1 Dimensions

An irrep of the group $M(1,1)$ can be extracted from GRR by fixing the sign of $p_0$ and eigenvalues of the operators $\hat{p}^2$, $\hat{S}$,

$$\hat{p}^2 f(x,\theta) = m^2 f(x,\theta), \tag{3.15}$$

$$\hat{S} f(x,\theta) = \lambda f(x,\theta), \tag{3.16}$$

where the chirality $\lambda$ distinguishes equivalent irreps labeled by identical eigenvalues $m^2$ of the Casimir operator $\hat{p}^2$. Solutions of this system have the form $f(x,\theta) = \psi(x)e^{i\lambda\theta}$, where $\hat{p}^2\psi(x) = m^2\psi(x)$.

According to (2.61), space reflection converts $e^{i\lambda\theta}$ to $e^{-i\lambda\theta}$. Irreps of the improper Poincaré group are labeled by mass $m$, sign $p_0$, intrinsic parity $n = \pm 1$, and spin $s = |\lambda|$ (as above, $s$ distinguishes equivalent irreps). In the rest frame, it is easy to write down functions with the mentioned characteristics:

$$e^{\pm imx^0}(e^{i\lambda\theta} \pm e^{-i\lambda\theta}). \tag{3.17}$$

States with arbitrary momentum can be obtained from (3.17) by hyperbolic rotations and form the basis of the unitary irrep of the improper group. On the other hand, the problem arises of constructing equations that, unlike the system (3.15)–(3.16), are invariant under the improper Poincaré group and have solutions with definite parity. These equations should combine states with chiralities $\pm\lambda$.

The general form of the equations linear in $\hat{p}^\mu$ is

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x,\theta) = \varkappa f(x,\theta), \tag{3.18}$$

where $\hat{\Gamma}^\mu = \hat{\Gamma}^\mu(\theta, \partial/\partial\theta)$. For invariance of (3.18) under space reflection $P$ and hyperbolic rotations, the operator $\hat{p}_\mu \hat{\Gamma}^\mu$ must commute with $P$ and $\hat{J} = \hat{L} + \hat{S}$, whence

$$\hat{\Gamma}^\mu \overset{P}{\to} (-1)^{\delta_{1\mu}} \hat{\Gamma}^\mu, \quad [\hat{\Gamma}^0, \hat{S}] = i\hat{\Gamma}^1, \quad [\hat{\Gamma}^1, \hat{S}] = i\hat{\Gamma}^0. \tag{3.19}$$

The operators

$$\hat{\Gamma}^0 = s\cosh\theta - \sinh\theta \frac{\partial}{\partial\theta}, \quad \hat{\Gamma}^1 = s\sinh\theta - \cosh\theta \frac{\partial}{\partial\theta}, \quad [\hat{\Gamma}^0, \hat{\Gamma}^1] = -i\hat{S} \tag{3.20}$$

obey these relations. One can construct the operators, which raise and lower chirality $\lambda$ by 1,

$$\hat{\Gamma}_+ = \hat{\Gamma}^0 - \hat{\Gamma}^1 = e^{-\theta}(s + \partial/\partial\theta), \quad \hat{\Gamma}_- = \hat{\Gamma}^0 - \hat{\Gamma}^1 = e^{\theta}(s - \partial/\partial\theta). \tag{3.21}$$

Operators $\hat{\Gamma}^0$, $\hat{\Gamma}^1$, and $\hat{\Gamma}^2 = -i\hat{S} = -\partial/\partial\theta$ obey the commutation relations of the generators of the $SO(2,1) \sim SU(1,1)$ group:

$$[\hat{\Gamma}^a, \hat{\Gamma}^b] = \epsilon^{abc}\hat{\Gamma}_c, \quad \hat{\Gamma}_a = \eta_{ab}\hat{\Gamma}^b, \quad \eta_{00} = \eta_{22} = -\eta_{11} = 1,$$
$$\hat{\Gamma}_a \hat{\Gamma}^a = s(s+1).$$

Thus, if symmetry with respect to space reflection holds, the condition of mass irreducibility (3.15) can be supplemented by Eq. (3.18) instead of (3.16). This means passing to a new set of commuting operators, namely from $\hat{p}_\mu$, $\hat{S}$ to $\hat{p}_\mu$, $\hat{p}_\mu \hat{\Gamma}^\mu$. Let us consider the system

$$\hat{p}^2 f(x,\theta) = m^2 f(x,\theta), \tag{3.22}$$

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x,\theta) = ms f(x,\theta). \tag{3.23}$$

The operator $\hat{S}$ does not commute with $\hat{p}_\mu \hat{\Gamma}^\mu$, and the particle with nonzero mass described by Eq. (3.23) cannot be characterized by certain chirality. In the rest frame, $p_0 = \pm m$, and the functions $f(x,\theta) = e^{\pm imx^0}\phi(\theta)$ should be eigenfunctions of operator $\hat{\Gamma}^0$ with eigenvalues $\pm s$. The equation

$$\hat{\Gamma}^0\phi(\theta) = [s\cosh\theta - (\sinh\theta)\partial/\partial\theta]\phi(\theta) = \varkappa\phi(\theta)$$

for $\varkappa = \pm s$ has solutions $[\cosh(\theta/2)]^{2s}$ and $[\sinh(\theta/2)]^{2s}$, respectively. We will consider two cases.

*Case 1.* The solutions of the system (3.22)–(3.23) are sought in the space of polynomials of $e^{-\theta/2}$ and $e^{\theta/2}$ that correspond to finite-dimensional nonunitary representations of $SU(1,1)$. Corresponding representations of the $SO(1,1)$ subgroup are also nonunitary. For these representations, the generator $\hat{S}$ is anti-Hermitian, and it is convenient to redefine the chirality operator as $i\hat{S}$. In the rest frame, a general solution of the system (3.22)–(3.23) is

$$f(x,\theta) = C_1 e^{imx^0}[\cosh(\theta/2)]^{2s} + C_2 e^{-imx^0}[\sinh(\theta/2)]^{2s}, \tag{3.24}$$

where $2s$ is a positive integer. Therefore, for a unique spin $s$, there are only two independent components (with positive and negative frequency). The space inversion takes $\theta$ to $-\theta$, and in the rest frame, solutions with different sign of $p_0$ and half-integer $s$ are characterized by opposite parity $\eta$. For integer $s$, all solutions are characterized by $\eta = 1$. Plane wave solutions, which correspond to a moving particle, can be obtained from (3.24) by a hyperbolic rotation by the angle $2\phi$:

$$\begin{aligned} f_{m,s}(x,\theta) = C_1 e^{ik_0x^0+ik_1x^1}\{\cosh[(\theta+\phi)/2]\}^{2s} \\ + C_2 e^{-ik_0x^0-ik_1x^1}\{\sinh[(\theta+\phi)/2]\}^{2s}, \end{aligned}$$

where $k_0 = m\cosh 2\phi$, $k_1 = m\sinh 2\phi$.

In the ultrarelativistic limit $\phi \to \pm\infty$, we have two states with chirality $\lambda = \pm s$, respectively. Thus, if in the rest frame, one may distinguish two components with positive and negative frequency, then in the massless limit, one may distinguish two components with positive and negative chirality.

The matrix form of the system (3.22)–(3.23) can be obtained by the decomposition of $f(x,\theta)$ over the basis $e^{\lambda\theta/2}$, $\lambda = -s, -s+1, \ldots, s$. There are $2s+1$ components $\psi(x)$ in this form, but only two of them are independent. Notice that representations of $SO(1,1)$ of the form $e^{\lambda\theta}$ are nonunitary for real $\lambda$ and the integral over $\theta$ is divergent. One can redefine the norm of a state with the help of the scalar product in the space of multicomponent functions $\psi(x)$, but this product is not positive definite.

For $s = 1/2$, substituting the function $f(x,\theta) = \psi_1(x)e^{\theta/2} + \psi_2(x)e^{-\theta/2}$ into Eq. (3.23), we obtain the two-dimensional Dirac equation (Abdalla *et al.*, 1991)

$$\hat{p}_\mu\gamma^\mu\Psi(x) = m\Psi(x), \quad \gamma^0 = \sigma_1, \quad \gamma^1 = -i\sigma_2, \quad 2\hat{S} = \gamma^3 = \sigma_3. \tag{3.25}$$

where $\Psi(x) = (\psi_1(x)\,\psi_2(x))^T$. The matrix $\gamma^3 = \gamma^0\gamma^1$ corresponds to the chirality operator and satisfies the condition $[\gamma^3, \gamma^\mu]_+ = 0$. On the other hand, this matrix corresponds to hyperbolic rotation, and similar to the $3+1$ case, one can write $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} - i\sigma^{\mu\nu}$, where $\sigma^{01} = i\gamma^3$. The invariant scalar product has the form $|\psi_1(x)|^2 - |\psi_2(x)|^2$.

For $s = 1$, substituting the function $f(x,\theta) = \psi_{11}(x)e^{\theta} + \psi_{12}(x) + \psi_{22}(x)e^{-\theta}$ into Eq. (3.23), we obtain

$$
\begin{aligned}
&(\hat{p}_\mu \hat{\Gamma}^\mu - m)\Psi(x) = 0,\\
&\Gamma^0 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}, \quad \Gamma^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & -1\\ 0 & 1 & 0\end{pmatrix},\\
&\hat{S} = \begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix},
\end{aligned}
\tag{3.26}
$$

where $\Psi(x) = (\psi_{11}(x)\,\psi_{12}(x)/\sqrt{2}\psi_{22}(x))^T$. Using (3.14) to convert spinor indices to vector ones, we obtain $\mathcal{F}_0 = \psi_{22} - \psi_{11}$, $\mathcal{F}_1 = \psi_{22} + \psi_{11}$, and $F_{01} = -\mathcal{F}_{10} = -i\psi_{12}$, and we obtain $p_0\mathcal{F}_1 - p_1\mathcal{F}_0 = -imF_{01}$, $ip_0F_{10} = m\mathcal{F}_1$, $ip_1F_{10} = m\mathcal{F}_0$. Thus, one can rewrite the $1+1$ "Duffin–Kemmer" equation (3.26) in the following form, which is similar to Proca equations in $3+1$ dimensions [see (5.85), (5.89)]:

$$
\partial_\mu \mathcal{F}_\nu - \partial_\nu \mathcal{F}_\mu = mF_{\mu\nu}, \quad \partial^\nu F_{\mu\nu} = m\mathcal{F}_\mu. \tag{3.27}
$$

As a consequence of (3.27), we obtain $\partial_\mu \mathcal{F}^\mu = 0$, $(\hat{p}^2 - m^2)\mathcal{F}^\mu = 0$. But the $1+1$-dimensional case is distinctly different from the $3+1$-dimensional case because the component $F_{01} = -F_{10}$ is characterized by zero chirality and thus the roles of $F_{\mu\nu}$ and $\mathcal{F}_\nu$ are interchanged.

In the massless case, the system (3.27) splits into two independent equations for the components $\mathcal{F}_\mu$ and $F_{\mu\nu}$, respectively,

$$
\partial_\mu \mathcal{F}_\nu - \partial_\nu \mathcal{F}_\mu = 0, \tag{3.28}
$$

$$
\partial^\mu F_{\mu\nu} = 0. \tag{3.29}
$$

The first equation has propagating solutions

$$
\mathcal{F}_0 = C_1 e^{ip(x^0+x^1)} + C_2 e^{ip(x^0-x^1)}, \quad \mathcal{F}_1 = C_1 e^{ip(x^0+x^1)} - C_2 e^{ip(x^0-x^1)}
$$

obeying the transversality condition $\partial_\mu \mathcal{F}^\mu = 0$. The second equation [free two-dimensional Maxwell equation (Abdalla *et al.*, 1991) corresponds to the components with zero chirality and has the trivial solution $F_{\mu\nu} = \text{const.}$ only. Notice that for real field $f^*(x,\theta) = f(x,\theta)$ components, $\mathcal{F}_\mu$ and $F_{\mu\nu}$ also are real, and propagating solutions do not exist for $m = 0$.

If, for $s = 1/2$ and $s = 1$, the first equation of the system (3.22)–(3.23) is the consequence of the second equation, then for $s > 1$, there are solutions of Eq. (3.23) with mass spectrum $m_i|s_i| = ms$, $s_i = s, s-1, \ldots, -s$. For the extraction of the improper Poincaré group characterized by certain mass $m$ and spin $s$ representations, it is necessary to use both equations of the system.

Notice that the chirality $\lambda$ of a particle described by (3.15)–(3.16) can be fractional, but the spin $s$ of a particle described by (3.22)–(3.23) can be only integer or half-integer for $m \neq 0$ and a finite number of components $\psi(x)$.

If $2s$ is not an integer, then acting by the raising operator on the state with lable $\lambda = -s$, we will not get to the state labeled by $\lambda = s$ and connected with the initial state by the space reflection; moreover, the spectrum of $\lambda$ is not bounded above. On the other hand, it is possible to develop an alternative approach (in particular, for massive particles with fractional spin) based on using infinite-dimensional unitary irreps of $SO(2, 1)$.

*Case 2.* Let us consider now the solutions of (3.22)–(3.23) in the space of square-integrable functions of $\theta$. In the rest frame, as we have seen above, there are two types of solutions. The solutions $[\sinh(\theta/2)]^{2s}$ are not square-integrable for any $s$ since the corresponding integral is divergent either at zero or at infinity. The solutions $[\cosh(\theta/2)]^{2s}$ for $s < 0$ are square-integrable:

$$\int_{-\infty}^{+\infty} [\cosh(\theta/2)]^{4s}\, d\theta = 2B(1/2, 2s).$$

Therefore, in the space of square-integrable functions, Eq. (3.23) has only positive-energy solutions. Solutions with $p_0 < 0$ correspond to the equation $\hat{p}_\mu \hat{\Gamma}^\mu f(x, \theta) = -msf(x, \theta)$. Normalized positive-energy solutions of the system (3.22)–(3.23) for a particle with spin $|s|$ and momenta $p_0 = m \cosh\phi$, and $p_1 = m \sinh\phi$ are

$$f(x, \theta) = (2\pi)^{-1}\, [2B(1/2, 2s)]^{-1/2}\, e^{ip_0x^0 + ip_1x^1}\{\cosh[(\theta + \phi)/2]\}^{-2|s|}. \quad (3.30)$$

In contrast to the case $d > 2$, solutions with distinct $s$ are nonorthogonal. The decomposition of the solutions (3.30) over the functions $e^{i\lambda\theta}$ [i.e., over $SO(1, 1)$ unitary irreps] corresponds to the Fourier integral expansion. We will consider properties of the positive-energy equations in more detail in the $2 + 1$-dimensional case below.

## 4. THREE-DIMENSIONAL CASE

### 4.1. Field on the Group $M(3)$

The case of the $M(3)$ group is characterized by many-dimensional spin space. On the other hand, the constructions allow a simple physical interpretation.

Using the operators $\hat{J}^i = \hat{L}^i + \hat{S}^i = (1/2)\epsilon^{ijk}\hat{J}_{jk}$, it is possible to rewrite the commutation relations (2.37) in the more compact form

$$[\hat{p}_i, \hat{p}_k] = 0, \quad [\hat{p}^i, \hat{J}^j] = i\epsilon^{ijk}\hat{p}_k, \quad [\hat{J}^i, \hat{J}^j] = i\epsilon^{ijk}\hat{J}_k. \quad (4.1)$$

The invariant measure on the group is given by the formulas

$$d\mu(x, \mathbf{z}) = Cd^3x\, \delta(|z_1|^2 + |z_2|^2 - 1)\, d^2z_1\, d^2z_2 = \frac{1}{16\pi^2} d^3x \sin\theta\, d\theta\, d\phi\, d\psi,$$

$$-\infty < x < +\infty, \quad 0 < \theta < \pi, \quad 0 < \phi < 2\pi, \quad -2\pi < \psi < 2\pi, \quad (4.2)$$

where

$$z_1 = \cos\frac{\theta}{2}e^{i(\psi-\phi)/2}, \quad z_2 = i\sin\frac{\theta}{2}e^{i(\psi+\phi)/2}$$

are the elements of the first column of matrix (2.46), $z^2 = -z_1, z^1 = z_2$, and $\theta, \phi, \psi$ are the Euler angles. The spin projection operators acting in the space of the functions on the group $f(x, \mathbf{z})$ have the form

$$\hat{S}_k = \frac{1}{2}\big(z\sigma_k\partial_z - \overset{*}{z}\overset{*}{\sigma}_k\,\partial_{\overset{*}{z}}\big), \quad z = (z^1\; z^2), \quad \partial_z = (\partial/\partial z^1\; \partial/\partial z^2)^T,$$

$$\hat{S}^{\mathrm{R}}_k = -\frac{1}{2}(\chi\overset{*}{\sigma}_k\partial_\chi - \overset{*}{\chi}\sigma_k\partial_{\overset{*}{\chi}}), \quad \chi = (z^1 - \overset{*}{\dot{z}}{}^2), \quad \partial_\chi = (\partial/\partial z^1 \; -\partial/\partial \overset{*}{\dot{z}}{}^2)^T. \quad (4.3)$$

In terms of Euler angles, we obtain

$$\hat{S}_3 = -i\partial/\partial\phi, \quad \hat{S}^{\mathrm{R}}_3 = i\partial/\partial\psi. \quad (4.4)$$

The operator $\hat{\mathbf{p}}^2$ and the operator of the spin projection on the direction of propagation $\hat{W} = \hat{\mathbf{p}}\hat{\mathbf{J}} = \hat{\mathbf{p}}\hat{\mathbf{S}}$ are Casimir operators. The eigenvalues $S(S+1)$ of the Casimir operator of the rotation subgroup in $z$-space, $\hat{\mathbf{S}}^2 = \hat{\mathbf{S}}^2_{\mathrm{R}}$, define spin $S$. Complete sets of the commuting operators $\{\hat{p}_k, \hat{W}, \hat{\mathbf{S}}^2, \hat{S}^3_{\mathrm{R}}\}$, $\{\hat{\mathbf{p}}^2, \hat{W}, \hat{\mathbf{J}}^2, \hat{S}_3, \hat{\mathbf{S}}^2, \hat{S}^{\mathrm{R}}_3\}$ consist of six operators (two Casimir operators, two left generators, and two right generators). The Casimir operator $\hat{W}$ does not commute with $\hat{L}_k$ and $\hat{S}_k$ separately, but only with the generators $\hat{J}_k = \hat{L}_k + \hat{S}_k$; therefore there are sets that do not include $\hat{W}$, for example, $\{\hat{p}^2, \hat{p}_3, \hat{L}_3, \hat{S}_3, \hat{\mathbf{S}}^2, \hat{S}^{\mathrm{R}}_3\}$ and $\{\hat{p}_\mu, \hat{S}_3, \hat{\mathbf{S}}^2, \hat{S}^{\mathrm{R}}_3\}$.

We will consider the first set, in this case eigenfunctions have the simplest form. This set includes two Casimir operators, the operator of spin squared $\hat{\mathbf{S}}^2$ and the generator $\hat{S}^{\mathrm{R}}_3$. The latter two generators commute with all left generators, but do not commute with right generators and label equivalent representations in the decomposition of the left GRR.

According to (4.4), the eigenfunctions of $\hat{S}^{\mathrm{R}}_3$, $\hat{S}^{\mathrm{R}}_3 \mid \cdots n\rangle = n \mid \cdots n\rangle$, have the form $\mid \cdots n\rangle = F(x, \theta, \phi)\exp(-in\psi)$ and differ only by a phase factor. As a consequence of the commutation relations of the generators $\hat{S}^{\mathrm{R}}_k$, the operators $\hat{S}^{\mathrm{R}}_\pm = \hat{S}^{\mathrm{R}}_1 \pm i\hat{S}^{\mathrm{R}}_2$ are the raising and lowering operators for the eigenfunctions of $\hat{S}^{\mathrm{R}}_3$,

$$\hat{S}^{\mathrm{R}}_\pm \mid \cdots n\rangle = C(S, n) \mid \cdots n \pm 1\rangle. \quad (4.5)$$

The intertwining operators $\hat{S}^{\mathrm{R}}_\pm$ consist of the generators of right rotations, which conserve the interval square according to (2.35). Moreover, the right rotations do not act on $x$. But there are no transformations (rotations and translations) of the reference frame, which connect representations with different $n$. Notice that the states labeled by $n$ and $-n$ are interchanged under charge conjugation (2.63).

The operator $\hat{\mathbf{S}}^2$ also labels equivalent representations of the $M(3)$ group. This operator commutes with all generators except right translations, and therefore an intertwining operator is a function of the latter generators. Right translations change both the interval and the spin. Therefore, it is natural to characterize a free particle in three-dimensional Euclidean space not only by momentum and spin projection on the direction of propagation, but also by spin $S$.

There are two standard realizations of the representation spaces, corresponding to eigenvalues $n = \pm 2S$ and $n = 0$ of the operator $\hat{S}_3^{\mathrm{R}}$.

The first realization is the space of analytic ($n = -2S$) functions $f(x, z)$ or antianalytic ($n = 2S$) functions $f(x, \overset{*}{z})$ of two complex variables $z^1, z^2$, $|z^1|^2 + |z^2|^2 = 1$, that is, the space of functions of two-component spinors. In particular, according to (4.3), for the space of analytic functions, we have

$$\hat{S}_k = \frac{1}{2} z\sigma_k \partial_z, \tag{4.6}$$

$\hat{S}_3^{\mathrm{R}} = -(z^1\partial/\partial z^1 + z^2\partial/\partial z^2)$ and $\hat{\mathbf{S}}^2 = \hat{S}_3^{\mathrm{R}}(\hat{S}_3^{\mathrm{R}} - 1)$. The eigenfunctions of the operator of spin squared are polynomials of the power $2S$ in $z^1, z^2$. The charge conjugation transformation connects equivalent irreps labeled by $n = \pm 2S$ and the spaces of analytic and antianalytic functions. This transformation reverses the direction of momentum and spin.

The second realization is the space of functions, which do not depend on the angle $\psi$, and corresponds to $n = 0$. It is the space of functions of five real variables on the manifold

$$\mathbb{R}^3 \times S^2, \quad d\mu = (4\pi)^{-1}\, d^3x \sin\theta\, d\theta\, d\phi.$$

The point in the spin space [i.e., on the sphere $S^2 \sim \mathbb{C}P^1 \sim SU(2)/U(1)$] can be defined by the spherical coordinates $\theta, \phi$ or by two complex variables

$$z_1 = \cos\frac{\theta}{2} e^{-i\phi/2}, \quad z^2 = \sin\frac{\theta}{2} e^{i\phi/2}$$

[in this case, one may use (4.6) for the spin projection operators], or by one complex number $z = z_1/z_2$ (this case corresponds to the realization in terms of the projective space $\mathbb{C}P^1$). In terms of variables $\theta, \phi$, the eigenfunctions of the operators $\hat{S}, \hat{S}_3$ are $P_S^s(\cos\theta)e^{is\phi}$, where $P_S^s(\cos\theta)$ are associated Legendre functions (Vilenkin, 1968).

Let us consider eigenfunctions of the set of the operators $\{\hat{p}_\mu, \hat{W}, \hat{\mathbf{S}}^2\}$ in the space of analytic function of $z^1, z^2$:

$$\begin{aligned} &\hat{p}_\mu f(x,z) = p_\mu f(x,z), \quad \hat{\mathbf{S}}^2 f(x,z) = S(S+1) f(x,z), \\ &\hat{\mathbf{p}}\hat{\mathbf{S}} f(x,z) = psf(x,z). \end{aligned} \tag{4.7}$$

The eigenfunctions of $\hat{\mathbf{S}}^2$ are polynomials of the power $2S$ in $z$ [the unitary irreps of $SU(2)$ are finite dimensional, therefore the spin $S$ and the spin projection on the direction of propagation $s$ are integer or half-integer]. Let $p_\mu = (0, 0, p)$; then the normalized solutions of the system (4.7) are

$$|0\,0pSs\rangle = (2\pi)^{-3/2}\left(\frac{(2S)!}{(S+s)!(S-s)!}\right)^{1/2}(z^1)^{S+s}(z^2)^{S-s}e^{ix_3p}.$$

The states with arbitrary direction of vector $p$ can be obtained by the rotation $P = UP_0U^\dagger$, $Z = UZ_0$, $P_0 = p\sigma_3$, $Z_0 = (z_1z_2)^T$:

$$|p_1p_2p_3Ss\rangle = (2\pi)^{-3/2}\left(\frac{(2S)!}{(S+s)!(S-s)!}\right)^{1/2} \times \left(z^1\overset{*}{u}_1 + z^2\overset{*}{u}_2\right)^{S+s}\left(-z^1u_2 + z^2u_1\right)^{S-s}e^{ipx}, \tag{4.8}$$

where $u_1, u_2$ are the elements of the first line of the matrix $U$. Notice that it is sufficient to use only two angles for the parametrization of matrix $U$ since the initial state has a stationary subgroup $U(1)$.

For the rest particle, $\hat{\mathbf{p}}^2 = \hat{\mathbf{p}}\hat{\mathbf{S}} = 0$, and only in this case are $M(3)$ irreps labeled by different $S$ nonequivalent.

In the general case, functions corresponding to $\alpha$ particle of spin $S$ have the form

$$f_S(x,z) = \sum_{n=0}^{2S}\phi^n(z)\psi_n(x), \quad \phi^n(z) = \left(C^n_{2S}\right)^{1/2}(z^1)^{S-n}(z^2)^n, \tag{4.9}$$

$$\int \overset{*}{f}_S(x,z)f'_{S'}(x,z)\ d\mu(x,z) = \delta_{SS'}\int\sum_{n=0}^{2S}\overset{*}{\psi}_n(x)\psi'_n(x)\ d^3x, \tag{4.10}$$

where $C^{2S}_n$ is the binomial coefficient and $d\mu(x,z)$ is the invariant measure (4.2). The relation (4.9) gives the connection between the description by the functions $f(x,z)$ and the standard description by the multicomponent function $\psi_n(x)$. It is easy to see that the action of the operators $\hat{S}_k = \frac{1}{2}z\sigma_k\partial_z$ on the function (4.9) reduces to the multiplication of the column $\psi(x)$ by $(2S+1)\times(2S+1)$ matrices $S_k$ of $SU(2)$ generators in the representation $T_S$, $\hat{S}_kf(x,z) = \phi(z)S_k\psi(x)$. The matrices $S_k$ obey the commutation relations of spin projection operators, namely $[S^i, S^j] = i\epsilon^{ijk}S_k$.

In particular, the linear function of $z^1, z^2$ corresponds to spin $S = 1/2$, and the action of the operators $\hat{S}_k$ on $\psi(x)$ is reduced to the multiplication by $\sigma$-matrices, $\hat{S}_kf(x,z) = \phi(z)\sigma_k\psi(x)$.

As mentioned above, the operator $\hat{\mathbf{S}}^2$ is not a Casimir operator of $M(3)$ and labels equivalent representations of the group. This operator is the direct analog of the Lorentz spin operator in the pseudo-Euclidean case, and we will consider its properties in detail.

1. The operator $\hat{\mathbf{S}}^2$ is composed of right generators commuting with all left generators and therefore is not changed under the coordinate

transformation (left transformations of the Euclidean group). The right transformations do not change the spin projection $s$ on the direction of propagation, but change both spin $S$ and interval (distance).

2. The operator $\hat{\mathbf{S}}^2$ does not depend on $x$ and commutes with the operators $x_k, \hat{p}_k, \hat{S}_k$; therefore, it is an integral of motion for any Hamiltonians of the form $\hat{H} = \hat{H}(x_k, \hat{p}_k, \hat{S}_k)$.
3. The eigenvalues of $\hat{\mathbf{S}}^2$ label irreps of the rotation subgroup in the spin space and define the possible values of the spin projection $s$.

Notice that in the representation theory of the Galilei group [symmetry group of nonrelativistic mechanics, which includes $M(3)$ and ensures more general description], irreps labeled by different eigenvalues of $\hat{\mathbf{S}}^2$ are not equivalent. The classification of irreps of the Galilei group can be based on the use of two invariant equations. The Schrödinger equation fixes the mass $m$, and the second equation fixes the eigenvalue of spin operator $\hat{\mathbf{S}}^2$ (Hamermesh, 1960; Levy-Leblond, 1963).

### 4.2. Field on the Group $M(2, 1)$ and Fractional Spin

Using the operators $\hat{J}^\rho = \hat{L}^\rho + \hat{S}^\rho = (1/2)\epsilon^{\rho\mu\nu}\hat{J}_{\mu\nu}$, it is possible to rewrite the commutation relations (2.37) in the form

$$[\hat{p}_\mu, \hat{p}_\nu] = 0, \quad [\hat{p}^\mu, \hat{J}^\nu] = -i\epsilon^{\mu\nu\eta}\hat{p}_\eta, \quad [\hat{J}^\mu, \hat{J}^\nu] = -i\epsilon^{\mu\nu\eta}\hat{J}_\eta. \tag{4.11}$$

The invariant measure on the group is given by the formulas (Vilenkin, 1968)

$$\begin{aligned} d\mu(x, \underline{z}) &= C\, d^3x\, \delta(|z_1|^2 - |z_2|^2 - 1)\, d^2z_1\, d^2z_2 = \frac{1}{8\pi^2} d^3x\, \sinh\theta\, d\theta\, d\phi\, d\psi, \\ &\quad -\infty < x < +\infty, \quad 0 < \theta < \infty, \quad 0 < \phi < 2\pi, \\ &\quad -2\pi < \psi < 2\pi, \end{aligned} \tag{4.12}$$

where $z_1 = \cosh\frac{\theta}{2}e^{i(\psi-\phi)/2}$, $z_2 = \sinh\frac{\theta}{2}e^{i(\psi+\phi)/2}$ are the elements of the first column of matrix $Z$, (2.46), and $\theta, \phi$, and $\psi$ are the analogs of the Euler angles; $z^2 = -z_1, z^1 = z_2$. The spin projection operators acting in the space of the functions on the group $f(x, \mathbf{z})$ have the form

$$\begin{aligned} \hat{S}^\mu &= \frac{1}{2}(z\gamma^\mu\partial_z - \overset{*}{z}\overset{*}{\gamma}^\mu\partial_{\overset{*}{z}}), \quad z = (z^1\, z^2), \quad \partial_z = (\partial/\partial z^1 \partial/\partial z^2)^T \\ \hat{S}^\mu_{\rm R} &= -\frac{1}{2}\left(\chi\overset{*}{\gamma}^\mu\partial_\chi - \overset{*}{\chi}\gamma^\mu\partial_{\overset{*}{\chi}}\right), \quad \chi = (z^1\ \overset{*}{z}{}^{\dot{2}}), \\ \partial_\chi &= \left(\partial/\partial z^1 \partial/\partial \overset{*}{z}{}^{\dot{2}}\right)^T, \end{aligned} \tag{4.13}$$

where $\gamma^\mu$ are three-dimensional $\gamma$-matrices,

$$\gamma^\mu = (\sigma_3, i\sigma_2, -i\sigma_1), \quad \gamma^\mu\gamma^\nu = \eta^{\mu\nu} - i\varepsilon^{\mu\nu\rho}\gamma_\rho. \tag{4.14}$$

Note that a nonequivalent set of $\gamma$-matrices, $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + i\varepsilon^{\mu\nu\rho}\gamma_\rho$, is used in some papers. In terms of the Euler angles, we obtain $\hat{S}^0 = -i\partial/\partial\phi$, $\hat{S}^0_{\mathrm{R}} = i\partial/\partial\psi$. The sets of commuting operators are the same as in the Euclidean case.

In consequence of the identity $\sigma_1 \overset{*}{U} \sigma_1 = U$, one can show that the matrix $\sigma_1$ is the invariant symmetric tensor converting dotted and undotted indices,

$$\overset{*}{z}_\alpha = (\sigma_1)_\alpha^{\dot\alpha} \overset{*}{z}_{\dot\alpha}. \tag{4.15}$$

According to (2.47), the invariant tensor $\sigma_{\mu\alpha\dot\alpha}$ connects a vector index and two spinor indices of different types. On the other hand, using the identity mentioned above, one can rewrite (2.47) in the form $x'^\nu(\sigma_\mu\sigma_1) = x^\mu U(\sigma_\mu\sigma_1)U^T$. Thus the invariant tensor, which we denote as

$$\breve{\sigma}_{\mu\alpha\beta} = (\sigma_\mu\sigma_1)_{\alpha\beta}, \quad \breve{\sigma}_{\mu\alpha\beta} = \breve{\sigma}_{\mu\beta\alpha}, \tag{4.16}$$

connects a vector index and two spinor indices of one type. Thus, one can write the generators $\hat{S}^\mu$ in the form $\hat{S}^\mu = \frac{1}{2}\breve{\sigma}^\mu_{\alpha\beta}(z^\alpha\partial^\beta + \overset{*}{z}^\alpha\overset{*}{\partial}^\beta)$. An analog of $\sigma^{\mu\nu}$-matrics in $2+1$ dimensions is $(\sigma^{\mu\nu})_{\alpha\beta} = \varepsilon^{\mu\nu\lambda}\breve{\sigma}_{\lambda\alpha\beta}$. Raising one of the spinor indices of $\breve{\sigma}_{\mu\alpha\beta}$, we obtain two sets of three-dimensional $\gamma$-matrices differing only by the signs of $\gamma^0$ and $\gamma^2$.

Similar to the Euclidean case, there are two standard realizations of the representation spaces, corresponding to eigenvalues $n = \pm 2S$ and $n = 0$ of the operator $\hat{S}^{\mathrm{R}}_3$.

The first realization is the space of analytic ($n = -2S$) functions $f(x, z)$ or antianalytic ($n = 2S$) functions $f(x, \overset{*}{z})$ of two complex variables $z^1, z^2$, $|z^2|^2 - |z^1|^2 = 1$, that is, the space of functions of two-component spinors. The eigenfunctions of $\hat{S}_\mu\hat{S}^\mu$ are homogeneous functions of degree $2S$ in $z$. According to (4.3), we have $\hat{S}^0_{\mathrm{R}} = -(z^1\partial/\partial z^1 + z^2\partial/\partial z^2)$ for the space of analytic functions and $\hat{S}^0_{\mathrm{R}} = \overset{*}{z}{}^{\dot 1}\partial/\partial\overset{*}{z}{}^{\dot 1} + \overset{*}{z}{}^{\dot 2}\partial/\partial\overset{*}{z}{}^{\dot 2}$ for the space of antianalytic functions. The eigenfunctions of $\hat{S}_\mu\hat{S}^\mu$ in these spaces are also eigenfunctions of $\hat{S}^0_{\mathrm{R}}$ with eigenvalues $n = \mp 2S$, respectively.

The second realization is the space of eigenfunctions of $\hat{S}^0_{\mathrm{R}}$ with zero eigenvalue. It is the space of functions of five real parameters on the manifold

$$\mathbb{R}^3 \times \mathbb{C}D^1, \quad d\mu = (2\pi)^{-1} d^3x \, \sinh\theta \, d\theta \, d\phi,$$

where $\mathbb{C}D^1 \sim SU(1,1)/U(1)$ is a complex disk. These functions do not depend on the angle $\psi$.

We recall some facts from the representation theory of $SU(1,1)$. For finite-dimensional nonunitary irreps $T^0_S$ of the $2+1$ Lorentz group $SU(1,1) \sim SO(2,1)$, the spin projection $s$ (the eigenvalue of $\hat{S}^0$) can be only integer or half-integer, $s = -S, \dots, S$, where $S \geq 0$. However, in $2+1$ dimensions, the Lorentz group does not have a compact non-Abelian subgroup. Therefore, there are infinite-dimensional unitary representations corresponding to fractional $S$. These representations are multivalued representations of $SU(1, 1)$. For single-valued representations of $SU(1, 1)$, the spin projection $s$ can be only integer or half-integer [for $SO(2, 1)$ only integer].

The representations of discrete series correspond to $S < -1/2$. Irreps of the positive discrete series $T_S^+$ are bounded by the lowest weight $s = -S$, irreps of the negative discrete series $T_S^-$ are bounded by the highest weight $s = S$, and irreps of the principal series correspond to $S = -1/2 + i\lambda$ and can be bounded by highest (lowest) weight only for $S = -1/2$. For other irreps of the principal series, the spectrum of $s$ is not bounded. Supplementary series correspond to $-1/2 < S < 0$ and are characterized by a nonlocal scalar product.

The weight diagrams of series on the plane $S$, $s$ are given visually in Gitman and Shelepin (1997) and Wybourne (1974).

Thus, there are only two possibilities for the description of a particle with fractional spin by means of unitary irreps of $SU(1, 1)$ with local scalar product. The first corresponds to the discrete or principal series irreps bounded by lowest (highest) weight, $|s| \geq |S| \geq 1/2$. The second corresponds to the principal series irreps which are not bounded.

Unitary irreps of discrete series are used for the description of anyons (Cortés and Plyushchay, 1994; Gitman and Shelepin, 1997; Jackiw and Nair, 1991; Plyushchay, 1992). Corresponding unitary infinite-component representations of $M(2, 1)$ were constructed (Cortés and Plyushchay, 1994; Jackiw and Nair, 1991; Plyushchay, 1992) in the space of functions of $x^\mu$ and the complex variable $z = z^1/z^2$, that is, on the coset space $M(2/1)/U(1)$. It was shown that RWE associated with irreps of the discrete series have solutions only with a definite sign of the energy. Thus, the mentioned RWE are analogs of Majorana equations in $3 + 1$ dimensions; this aspect is considered in more detail in Cortés and Plyushchay (1994). Various formulations of the higher spin theory based on finite-component representations were considered, in particular, in Deser and Kay (1983), Deser (1984), Aragone and Deser (1984), Gitman and Tyutin (1997), and Vasiliev and Tyutin (1997).

### 4.3. Relativistic Wave Equations in 2 + 1 Dimensions

Let us fix the eigenvalues of the Casimir operators of the Poincaré group and of the spin Lorentz subgroup:

$$\hat{p}^2 f(x, \mathbf{z}) = m^2 f(x, \mathbf{z}), \tag{4.17}$$

$$\hat{p}_\mu \hat{S}^\mu f(x, \mathbf{z}) = K f(x, \mathbf{z}), \tag{4.18}$$

$$\hat{S}_\mu \hat{S}^\mu f(x, \mathbf{z}) = S(S + 1) f(x, \mathbf{z}). \tag{4.19}$$

Below we will call the operator $\hat{S}_\mu \hat{S}^\mu$ the operator of the Lorentz spin square.

Equations (4.17) and (4.18) define a subrepresentation of the left GRR of $M(2, 1)$, which is characterized by mass $m$, Lorentz spin $S$, and the eigenvalue $K$ of the Lubanski–Pauli operator. For $m = 0$, we suppose $K = 0$, which is true for irreps with a finite number of spinning degrees of freedom. The general cases for $m = 0$ and for $m$ imaginary were discussed in Binegar (1982) and Gitman and Shelepin (1997).

Possible values of $K$ can be easily described in the massive case. Here we can use a rest frame where $\hat{p}_\mu \hat{S}^\mu = \hat{S}^0 m \,\text{sign} p_0$. Thus, $K = sm = s^0 m$ for $p_0 > 0$ and $K = sm = -s^0 m$ for $p_0 < 0$, where $s^0$ is the eigenvalue of $\hat{S}^0$. The spectrum of $\hat{S}^0$ depends on the representation of the Lorentz group.

The variable $s$ labels irreps of the group $M(2, 1)$ and can take both positive and negative values. Thus, there exits an analogy with massless particles in $3 + 1$ dimensions characterized by helicity. In both cases, $SO(2)$ is the little group, and single-valued irreps of $SO(2)$ are labeled by integer number $2s$. [It is a particular case of the connection between the massive fields in $d$ dimensions and massless fields in $d + 1$ dimensions (Aragone *et al.*, 1987; Vasiliev and Tyutin, 1997). Therefore, we will call $s$ the helicity and $|s|$ the spin.

Corresponding to (2.61), space reflection reduces to rotation by $\pi$ around the axis $x^0$ and converts $Z$ to $(Z^\dagger)^{-1} = \sigma_3 Z \sigma_3$, or $z_1 \to z_1, z_2 \to -z_2$. The operators $\hat{p}^0, \hat{S}^0$ do not change. Thus, distinct from the $(3 + 1)$-dimensional case, space reflection leaves helicity unaltered.

Fixing $S$ in (4.19), we pass to the space of homogeneous functions of degree $2S$ in $z_1, z_2$. According to the sign of $S$, below we will consider two possible choices of $SU(1, 1)$ irreps bounded either on both sides or on one side, respectively.

Finite-dimensional nonunitary irreps $T_S^0$ of $SU(1, 1)$ are labeled by positive integer or half-integer $S$. The basis in the representation space is formed by polynomials of power $2S$ in $z$; see (A2). We denote corresponding representations of $M(2, 1)$ as $T_{m,s}^0$.

Infinite-dimensional unitary irreps $T_S^-$ $(T_S^+)$ of $SU(1, 1)$ are labeled by negative $S < -1/2$ and are bounded by the highest (lowest) weight. The basis in the representation space is formed by quasipolynomials of power $2S$ in $z$; see (A3). We denote corresponding representations of $M(2, 1)$ as $T_{m,s}^-$ $(T_{m,s}^+)$.

One can represent the function $f(x, z)$ in the form

$$f(x, z) = \phi(z)\psi(x), \tag{4.20}$$

where $\phi(z)$ is a line composed of the basis elements $\phi_n(z)$ of the corresponding $SU(1, 1)$ representation, and $\psi(x)$ is a column composed of the coefficients in the decomposition over this basis. The action of the differential operators $\hat{S}^\mu$ on a function $f(x, z)$ may be presented in terms of matrices

$$\hat{S}^\mu f(x, z) = \phi^n(z)(S^\mu)_n^{n'} \psi_{n'}(x), \tag{4.21}$$

where $S^\mu$ are $SU(1, 1)$ generators in the representation $T_S$ [see Appendix A and also Gitman and Shelepin (1997). They obey the commutation relations of the $SU(1, 1)$ group $[S^{\mu}, S^{\nu}] = -i\epsilon^{\mu\nu\eta}S_{\eta}$.

For fixed $S$ in the matrix representation, Eqs. (4.17) and (4.18) have the form

$$(\hat{p}^2 - m^2)\psi(x) = 0, \tag{4.22}$$

$$\big(\hat{p}_{\mu}S^{\mu} - sm\big)\psi(x) = 0, \tag{4.23}$$

According to (4.23),

$$\psi^{\dagger}(x)\big(S^{\dagger\mu}\ \overleftarrow{\hat{p}_{\mu}} + sm\big) = 0.$$

It follows from the explicit expressions (A4) that for $T^{0}_{m,s}$ the relation $S^{\dagger\mu} = \Gamma S^{\mu}\Gamma$, where the relations $(\Gamma)_{nn'} = (-1)^n\delta_{nn'}$ and $\Gamma^2 = 1$ hold. For $T^{+}_{m,s}$ and $T^{-}_{m,s}$, the matrices $S^{\mu}$ are Hermitian, $S^{\dagger\mu} = S^{\mu}$, according to (A5). Let us introduce the notation

$$\bar{\psi} = \psi^{\dagger}\Gamma \quad \text{for } T^{0}_{m,s},$$

$$\bar{\psi} = \psi^{\dagger} \qquad \text{for } T^{+}_{m,s}, T^{-}_{m,s}.$$

The function $\bar{\psi}(x)$ obeys the equation

$$\bar{\psi}(x)(S^{\mu}\overleftarrow{\hat{p}_{\mu}} + sm) = 0. \tag{4.24}$$

As a consequence of the relations $S^{\dagger\mu} = \Gamma S^{\mu}\Gamma$ and $(S^{\mu})^{\dagger} = -(-1)^{\delta_{0\mu}}S^{\mu}$, we obtain that for irrep $T^{0}_{m,s}$ finite, the transformation matrices obey the equation $\Gamma T^{\dagger}(g)\Gamma = T^{-1}(g)$. Therefore, $\bar{\psi}(x)\psi(x)$ is a scalar density, and one can define a scalar product in the space of columns

$$(\psi'(x), \psi(x)) = \int \bar{\psi}'(x)\psi(x)\, d^3x. \tag{4.25}$$

The scalar density is positive definite for $T^{+}_{m,s}$ and $T^{-}_{m,s}$, in contrast to the case of $T^{0}_{m,s}$.

As a consequence of (4.23) and (4.24), the continuity equation holds,

$$\partial_{\mu}j^{\mu} = 0, \quad j^{\mu} = \bar{\psi}S^{\mu}\psi. \tag{4.26}$$

Together with the current vector $j^{\mu}$, by analogy with the four-dimensional case (Gel'fand *et al.*, 1963), one can associate with the linear equation (4.23) the energy-momentum tensor $T^{\mu\nu}$ and the energy density $W = -T^{00}$:

$$T^{\mu\nu} = \mathrm{Im}\left(S^{\mu}\frac{\partial\psi}{\partial x^{\nu}}, \psi\right), \quad W = -T^{00} = -\mathrm{Im}\left(S^{0}\frac{\partial\psi}{\partial x^{0}}, \psi\right). \tag{4.27}$$

If the matrix $S^0$ is diagonall, then the positiveness of $W(x)$ is equivalent to the requirement that

$$(S^0\psi, S^0\psi) \geq 0 \tag{4.28}$$

for all vectors $\psi$ (Gel'fand *et al.*, 1963). In particular, for $T^{+}_{m,s}$ and $T^{-}_{m,s}$ the relation $(S^0\psi, S^0\psi) = \psi^\dagger S^0 S^0 \psi \geq 0$ holds, and the energy density is positive definite.

There are two cases when Eq. (4.22) is the consequence of (4.23). Indeed, multiplying Eq. (4.18) by $\hat{p}_\mu S^\mu + ms$, one gets

$$(\hat{p}_\mu S^\mu + ms)(\hat{p}_\nu S^\nu - ms)\psi(x) = \left(\frac{1}{2}\hat{p}_\mu \hat{p}_\nu [S^\mu, S^\nu]_+ - m^2 s^2\right)\psi(x) = 0. \tag{4.29}$$

In the particular case $S = 1/2$, we have $s = \pm 1/2$, $S^\mu = \gamma^\mu/2$, and (4.29) is merely the Klein–Gordon equation (4.22). In the general case, the matrices $S^\mu$ are not $\gamma$-matrices in higher dimensions, and the squared equation (4.29) does not coincide with the Klein–Gordon equation (4.22). Using the rest frame, one can show that Eq. (4.22) follows from (4.23) also in the case of the vector representation $S = 1$, $s = \pm 1$. In the other cases, for the identification of the irrep of $M(2, 1)$, it is necessary to use both equations of the system (4.22), (4.23). Notice that another approach to the description of fields with fixed spin and mass was suggested in Plyushchay (1997), and this approach is based on the system of spinor linear equations.

It is naturaly to connect the spin value with the highest (lowest) weight of the irrep of the Lorentz group, $s = \pm S$. This means that up to a sign (plus for $p_0 > 0$, minus for $p_0 < 0$), $s$ is equal to the maximal or minimal eigenvalue of the operator $\hat{S}^0$ in the representation $T_S$ of the Lorentz group. According to (4.17)–(4.19), in this case, the functions $f(x, z)$ obey

$$\hat{p}^2 f(x, \mathbf{z}) = m^2 f(x, \mathbf{z}), \tag{4.30}$$

$$\hat{p}_\mu \hat{S}^\mu f(x, \mathbf{z}) = msf(x, \mathbf{z}), \quad s = \pm S, \tag{4.31}$$

$$\hat{S}_\mu \hat{S}^\mu f(x, \mathbf{z}) = S(S+1) f(x, \mathbf{z}). \tag{4.32}$$

In the framework of the system (4.30)–(4.32), *there are two possibilities to describe one and the same spin*:

1. Equations for $f(x, \mathbf{z}) = \phi(\mathbf{z})\psi(x)$, where $\phi(\mathbf{z})$ transforms under the finite-dimensional nonunitary irrep of the Lorentz group.
2. Equations for $f(x, \mathbf{z}) = \phi(\mathbf{z})\psi(x)$, where $\phi(\mathbf{z})$ transform under the infinite-dimensional unitary irrep of the Lorentz group. These equations allow us to describe also particles with fractional spin (anyons).

*Case 1*. First, consider the Poincaré group representations $T^0_{m,s}$ associated with *finite-dimensional nonunitary irreps* of $SU(1, 1)$. In this case, $S$ has to be positive, integer or half-integer. In the rest frame, the solutions of the system (4.30)–(4.32) in the space of analytic functions (polynomials of power $2S$

in $z^1, z^2$) are

$$s = S > 0: \quad f(x,z) = C_1(z^1)^S e^{imx^0} + C_2(z^2)^S e^{-imx^0}, \tag{4.33}$$

$$s = -S < 0: \quad f(x,z) = C_3(z^1)^S e^{-imx^0} + C_4(z^2)^S e^{imx^0}. \tag{4.34}$$

For unique mass and spin, there exist four independent components differing in the signs of $p_0$ and $s$, which correspond to four irreps of $M(2,1)$. The separation by the sign of the helicity $s$ is absolute since these states are solutions of different equations. But states with different sign of $p_0$ are solutions of one and the same equation. Hence, the energy spectrum of solutions is not bounded below or above.

In the space of antianalytic functions (polynomials of power $2S$ in $\overset{*}{z}{}^{\dot 1}, \overset{*}{z}{}^{\dot 2}$), the solutions of the system (4.30)–(4.32) are

$$s = S > 0: \quad f(x,\overset{*}{z}) = C_1(\overset{*}{z}{}^{\dot 1})^S e^{-imx^0} + C_2(\overset{*}{z}{}^{\dot 2})^S e^{imx^0},$$

$$s = -S < 0: \quad f(x,\overset{*}{z}) = C_3(\overset{*}{z}{}^{\dot 1})^S e^{imx^0} + C_4(\overset{*}{z}{}^{\dot 2})^S e^{-imx^0}.$$

These solutions are connected with the previous case (4.33), (4.34) by a charge conjugation (2.63) and therefore may be treated as the solutions describing antiparticles.

The wave function (4.33) corresponding to the helicity $s = -S$ has the form $C(z^2)^{2S} e^{ip_0x^0}$, $p_0 = m$, in the rest frame. Acting on it by finite transformations, we get a solution in the form of the plane wave, which is characterized by the momentum $p$:

$$P = UP_0U^\dagger, \quad P_0 = mI, \quad Z = UZ_0, \quad Z_0 = (z_1\, z_2)^T,$$

$$f(x,z) = (2\pi)^{-3/2}(z^2u_1 - z^1u_2)^{2S} e^{ipx}. \tag{4.35}$$

The state with $P_0 = mI$ has the stationary subgroup $U(1)$, and we can take elements $u_1 = \cosh(\theta/2)$ and $u_2 = \sinh(\theta/2)e^{i\omega}$ of the first line of the matrix $U$, which depends on two parameters only. Thus, $p_0 = E = m\cosh\theta$ and $-p_1 + ip_2 = m\sinh\theta\, e^{i\omega}$, and one can express the parameters $u_1$ and $u_2$ via the momentum $p$:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} E+m \\ -p_1 + ip_2 \end{pmatrix}. \tag{4.36}$$

The $2S+1$ components $\psi_n(x)$ are the coefficients in the decomposition of the function (4.35) over the basis $\phi^n(z)$, $f(x,z) = \phi^n(z)\psi_n(x)$, $n = 0, 1, \ldots, 2S$:

$$\psi_n(x) = (2\pi)^{-3/2}\big(C_{2S}^n\big)^{1/2}(u_1)^{2S-n}(-u_2)^n e^{ipx}$$

$$= (2\pi)^{-3/2}\big(C_{2S}^n\big)^{1/2} \frac{(E+m)^{2S-n}(p_1 - ip_2)^n}{(2m(E+m))^S} e^{ipx}. \tag{4.37}$$

In the particular case $S = 1/2$, we get

$$\psi(x) = \frac{1}{\sqrt{2m(E-m)}}\begin{pmatrix} p_2 - ip_1 \\ E+m \end{pmatrix} e^{ipx}.$$

Considering the system (4.31)–(4.32) without the condition of mass irreducibility (4.30), it is easy to see that the charge density $j^0 = \psi^\dagger \Gamma S^0 \psi$ is positive definite only for $S = 1/2$, and the energy density $-T^{00}$ is positive definite only for $S = 1$. The scalar density $\bar{\psi}\psi = \psi^\dagger \Gamma \psi$ is not positive definite.

Let us show that for particles with half-integer spin described by the system (4.30)–(4.32), the charge density $j^0$ of (4.26) is positive definite. In the rest frame, solutions of the system (4.30)–(4.32) have only two components (labeled by $s_0 = \pm S$), which we denote as $\psi_S(x)$ and $\psi_{-S}(x)$. For half-integer spin, the inequality $j^0 = \psi^\dagger \Gamma S^0 \psi = S(|\psi_S|^2 + |\psi - s|^2) > 0$ holds. For $S \geq 3/2$, from the explicit form of the matrices $S^1$ and $S^2$ of (A4), one can obtain that in the rest frame, $j^1 = j^2 = 0$, and therefore the square of the current vector $(j^0)^2 - (j^1)^2 - (j^2)^2$ is positive. Therefore, $j^0 > 0$ for any plane wave.

Thus, the charge density $j^0$ is positive definite for half-integer-spin particles described by representations $T^0_{m,s}$ of $M(2,1)$. The scalar density and the energy density are proportional to $\psi^\dagger \Gamma \psi = |\psi_S|^2 - |\psi_{-S}|^2$ in the rest frame and therefore are indefinite.

Let us consider now particles with integer spin. In the rest frame, the solutions of the system also have only two components, $\psi_S(x)$ and $\psi_{-S}(x)$, $(S^0\psi, S^0\psi) = \psi^\dagger \Gamma S^0 S^0 \psi = S^2(|\psi_S|^2 + |\psi_{-S}|^2) > 0$. Thus, the energy density is positive definite for integer-spin particles described by representations $T^0_{m,s}$ of $M(2,1)$. The charge density is proportional to $|\psi_S|^2 - |\psi_{-S}|^2$ in the rest frame and therefore is indefinite.

Consider two particular cases explicitly. For $S = 1/2$, the decomposition (4.20) has the form

$$f(x,z) = z^1\psi_1(x) + z^2\psi_2(x), \quad \psi'(x') = U^{-1}\psi(x), \quad \psi(x) = (\psi_1(x)\,\psi_2(x))^T. \tag{4.38}$$

Taking into account the relation $U^{-1} = \sigma_3 U^\dagger \sigma_3$, which is valued for the $SU(1,1)$ matrices, we get the transformation law for the line $\bar{\psi} = \psi^\dagger \sigma_3$, $\bar{\psi}'(x') = \bar{\psi}(x)U$. The product $\bar{\psi}(x)\psi(x) = |\psi_1(x)|^2 - |\psi_2(x)|^2$ is the scalar density.

Thus, in the case under consideration, we have two equivalent descriptions, one in terms of functions (4.38) and the other in terms of lines $\bar{\psi}(x)$ or columns $\psi(x)$. One can find the action of the operators $\hat{S}^\mu$ in the latter representation, and Eq. (4.23) can be rewritten in the form of the 2 + 1 Dirac equation

$$\hat{S}^\mu \psi(x) = \frac{1}{2}\gamma^\mu \psi(x), \quad (\hat{p}_\mu \gamma^\mu \mp m)\psi(x) = 0, \tag{4.39}$$

where minus corresponds to $s = 1/2$, plus corresponds to $s = -1/2$, and $\gamma^\mu$ are

$2 \times 2$ $\gamma$-matrices (4.14) in $2+1$ dimensions. The functions $\psi = (\psi^1\, 0)^T$ and $\psi = (0\, \psi^2)^T$ are eigenvectors of the operator $\hat{S}^0$ with the eigenvalues $\pm 1/2$.

Sometimes two equations for $s = \pm 1/2$ are written as one equation for the four-component reducible representation (Gitman and Tyutin, 1997; Vshivtsev *et al.*, 1998), $(\hat{p}_\mu \Gamma^\mu - m)\Psi(x) = 0$, where $\Gamma^\mu = \mathrm{diag}(\gamma^\mu, -\gamma^\mu)$ which corresponds to the simultaneous consideration of particles with opposite helicities.

For $S = 1$, the decomposition (4.20) has the form

$$f(x, z) = \psi_{11}(x)(z^1)^2 + \psi_{12}(x) z^1 z^2 + \psi_{22}(x)(z^2)^2, \tag{4.40}$$

where $\psi(x) = (\psi_{11}(x)\, \psi_{12}(x)/\sqrt{2}\, \psi_{22}(x))^T$ is subjected to Eq. (4.23) with the matrices

$$S^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S^1 = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$S^2 = -\frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{4.41}$$

If, instead of the cyclic components $\psi_{\alpha\beta}(x)$, one introduces new (Cartesian) components $\mathcal{F}_\mu = \check{\sigma}_\mu^{\alpha\beta} \psi_{\alpha\beta}(x)$, where $\check{\sigma}_{\mu\alpha\beta}$ is defined in (4.16) $\mathcal{F}_0 = -2\psi^{12}$, $\mathcal{F}_1 = \psi^{11} + \psi^{22}$, $\mathcal{F}_2 = i(\psi^{22} - \psi^{11})$, then Eq. (4.23) takes the form (Gitman and Shelepin, 1997)

$$\partial_\mu \varepsilon^{\mu\nu\eta} \mathcal{F}_\eta - sm\mathcal{F}^\nu = 0. \tag{4.42}$$

A transversality condition follows from (4.42), $\partial_\mu \mathcal{F}^\mu = 0$. One can see now that Eq. (4.42) are in fact field equations of the so-called “self-dual” free massive field theory (Townsend *et al.*, 1984). As remarked in Deser and Jackiw (1984), this theory is equivalent to the topologically massive gauge theory with the Chern–Simons term (Deser *et al.*, 1982). Indeed, the transversality condition allows us to introduce gauge potentials. $A_\mu$, namely a transverse vector can be written as a curl $\mathcal{F}^\mu = \varepsilon^{\mu\nu\lambda} \partial_\nu A_\lambda = \varepsilon^{\mu\nu\lambda} F_{\nu\lambda}/2$, where $F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu$ is the field strength. Thus, $\mathcal{F}^\mu$ appears to be the dual field strength, which is a three-component vector in $2+1$ dimensions. Then (4.42) implies the following for $F_{\mu\nu}$:

$$\partial_\mu F^{\mu\nu} - \frac{sm}{2} \varepsilon^{\nu\alpha\beta} F_{\alpha\beta} = 0,$$

which represents the field equations of the topologically massive gauge theory.

To describe a neutral spin-1 particle coinciding with its antiparticle, we consider the function

$$f(x, \mathbf{z}) = \psi_{11}(x) z^1 \overset{*}{z}{}^1 + \psi_{12}(x)(z^1 \overset{*}{z}{}^2 + \overset{*}{z}{}^1 z^2)/2 + \psi_{22}(x) z^2 \overset{*}{z}{}^2, \tag{4.43}$$

where we have used (4.15) for the conversion to undotted indices. The spin part of the function (4.43) depends not on three angles as in the case (4.40), but on two angles only. This function is an eigenfunction of the operator $\hat{S}^3_{\text{R}}$ with zero eigenvalue. Substituting (4.43) into (4.31), we again obtain Eq. (4.42).

*Case 2*. Consider now Poincaré group representations $T^+_{m,s}$ and $T^-_{m,s}$ associated with *unitary infinite-dimensional irreps* of $SU(1,1)$ with highest (lowest) weight. In this case, $S$ can be noninteger, $S < -1/2$ (discrete series) or $S = -1/2$ (principal series). Eigenvalues of $\hat{S}^0$ can take only positive values for discrete positive series, $s^0 = -S + n$, and only negative valuses for negative one, $s^0 = S - n$, where $n = 0, 1, 2, \ldots$.

Let us consider the energy spectrum of the system (4.30)–(4.32) for $m \neq 0$. According to the first equation, $p_0 = \pm m$. The second equation ensures the relation between spectra of the operators $\hat{p}_0$ and $\hat{S}^0$,

$$p_0 s^0 = ms. \tag{4.44}$$

For representations $T^0_{m,s}$, which correspond to finite-dimensional irreps $T^0_S$ of the Lorentz group, the value of $s^0$ can be positive or negative. Therefore, for any $s$, there exist both positive-frequency and negative-frequency solutions, and the representations $T^0_{m,s}$ split into two irreps characterized by sign $p_0 = \pm 1$.

For unitary $SU(1,1)$ irreps with highest (lowest) weight, the spectrum of $\hat{S}^0$ has a definite sign. For $T^+_S$, which act in the space of analytic functions, the spectrum of the operator $\hat{S}^0$ is positive, and for $T^-_S$, which act in the space of antianalytic functions, it is negative. Therefore, the sign of energy $p_0$ coincides with the sign of $s$ for $T^+_S$ and the signs of $p_0$ and $s$ are opposite for $T^-_S$. Thus, $T^+_{m,s}$ and $T^-_{m,s}$ are irreps of $M(2,1)$.

As in the case of the representations $T^0_{m,s}$, for unique mass and spin there are four states distinguished by the signs of $p_0$ and $s$. In the rest frame, there are two solutions of the system in the space of analytic functions:

$$p_0 > 0, \quad s > 0: \quad f(x,z) = (2\pi)^{-3/2}(z^2)^S e^{imx^0}, \tag{4.45}$$

$$p_0 < 0, \quad s < 0: \quad f(x,z) = (2\pi)^{-3/2}(z^2)^S e^{-imx^0}. \tag{4.46}$$

The solutions are connected by time reflection $T'$ (2.62). In the space of antianalytic functions, there are also two solutions:

$$p_0 > 0, \quad s < 0: \quad f(x,\overset{*}{z}) = (2\pi)^{-3/2}(\overset{*}{\dot{z}}{}^2)^S e^{imx^0} \tag{4.47}$$

$$p_0 < 0, \quad s > 0: \quad f(x,\overset{*}{z}) = (2\pi)^{-3/2}(\overset{*}{\dot{z}}{}^2)^S e^{-imx^0}. \tag{4.48}$$

These solutions are connected, respectively, with (4.45) and (4.46) by Schwinger time reversal $T_{Sch} = CT'$, which turns particles into antiparticles. Thus, there exist four equations distinguished by the sign of $s$ and by the used functionalspace (irrep

$T_S^+$ or $T_S^-$ of the Lorentz group), and any equation has solutions only with a definite sign of $p_0$.

In contrast to the case of $T^0_{m,s}$, where the energy spectrum $p_0$ is not bounded both above and below, the energy spectrum has a definite sign. In any inertial frame, the spectrum is bounded below by $p_0 = m$ for the solutions (4.45) and (4.47) and above by $p_0 = -m$ for the solutions (4.46) and (4.48).

For the unitary irreps of $M(2, 1)$ under consideration, which correspond to the irreps of the discrete series of the Lorentz group, integration of the functions (A3) in the invariant measure (4.12) gives

$$\int \overset{*}{f}_{S_1}(x, \mathbf{z})\, f'_{S_2}(x, \mathbf{z})\, d\mu(x, \mathbf{z}) = \delta_{S_1 S_2} \int \sum_{n=0}^{\infty} \psi^n(x)\psi'^n(x)\, d^3x,$$

$$\int \overset{*}{f}_{S_1}(x, \mathbf{z})\, f'_{S_2}(x, \mathbf{z})\, d\mu(\mathbf{z}) = \delta_{S_1 S_2} \psi^\dagger(x)\psi'(x), \tag{4.49}$$

In particular, the states (4.45)–(4.48) have the norm $\delta_{SS'}\delta(p - p')$. For the principal series, $j = -1/2 + i\lambda$, and $\delta_{j_1 j_2}$ in (4.49) is changed by $\delta(\lambda_1 - \lambda_2)$. At the same time, the integral over the spin space diverges for the representations $T^0_{m,s}$, which correspond to finite-dimensional irreps of the Lorentz group.

Arbitrary plane wave solutions can be obtained by analogy with the above case of $T^0_{m,s}$. For example, for the states (4.45), one can get the formula (4.37), where now $C^n_{2S}$ are the coefficients from (A3) and $n = 0, 1, 2, \ldots$. The power $2S$ is negative, and the decomposition $f(x, z) = \phi_n(z)\psi^n(x)$ contains an infinite number of terms.

Let us summarize some properties of the unitary irreps under consideration. Irreps $T^+_{m,s}$ and $T^-_{m,s}$ of the Poincaré group describe particles and antiparticles, respectively. The charge density $j^0 = \psi^\dagger S^0 \psi$ is positive definite for particles and negative definite for antiparticles. The energy density is positive definite in both cases since $(S^0\psi, S^0\psi) = \psi^\dagger S^0 S^0 \psi > 0$. For the unitary irreps, the scalar density $\psi^\dagger\psi$ is also positive definite, in contrast to the finite-dimensional case. The existence of positive-definite scalar density ensures the possibility of a probability amplitude interpretation of $\psi(x)$.

Thus, in $2 + 1$ dimensions, the problem of the construction of *positive-energy RWEs* is solved by the system (4.30)–(4.32) for the infinite-dimensional unitary irreps $T^+_{m,s}$ (signs of $p_0$ and $s$ are the same) or $T^-_{m,s}$ (signs of $p_0$ and $s$ are opposite) characterized by the mass $m$ and the helicity $s$. These irreps of the Poincaré group are associated with irreps $T_S^+$ and $T_S^-$ of the Lorentz group with lowest (highest) weight. Charge conjugation, changing signs of $p_0$ and $s^0$, leaves the helicity $s$ invariant and turns $T^+_{m,s}$ into $T^-_{m,s}$.

An interesting problem is to find an explicit form of the intertwining operator $A$ for the unitary irreps $T^+_{m,s}$, $T^-_{m,s}$ and the representation $T^0_{m,s}$ labeled by the same mass $m$ and helicity $s$, but associated with finite-dimensional nonunitary irreps of the Lorentz group, $AT^0_{m,s} = (T^+_{m,s} \oplus T^-_{m,s})A$. The intertwining operator is nonunitary and must be a function of the generators of right translations, since other generators commute with the Lorentz spin square operator $\hat{S}_\mu \hat{S}^\mu$ and cannot change the representation of the spin Lorentz subgroup.

Notice that the $2 + 1$ Dirac equation arises also in the case of unitary infinite-dimensional irreps $T^+_S$ and $T^-_S$ of the Lorentz group not as an equation for a true wave function, but as an equation for spin-coherent state evolution. In this case, the equation includes effective mass $m_s = |s/S|m$, $s = -S, -S+1, \ldots$ (Gitman and Shelepin, 1997).

Among the above RWE are those that describe particles with fractional real spin. These equations are associated with unitary multivalued irreps of the Lorentz group and can be used to describe anyons.

In spite of the fact that the number of independent polarization states for a massive $2 + 1$ particle is one, the vectors of the corresponding representation space of irreps $T^+_{m,s}$, $T^-_{m,s}$ have an infinite number of components in the matrix representation. Thus, the $z$-representation is more convenient in this case.

## 5. FOUR-DIMENSIONAL CASE

### 5.1. Field on the Group $M(3,1)$

The generators and the action of the left GRR on the functions $f(x, \mathbf{z})$ are given by formulas (2.37) and (2.44). For spin projection operators, it is convenient to use the three-dimensional vector notation $\hat{S}_k = \frac{1}{2}\epsilon_{ijk}\hat{S}^{ij}$, $\hat{B}_k = \hat{S}_{0k}$. An explicit calculation gives

$$\hat{S}_k = \frac{1}{2}(z\sigma_k\partial_z - \overset{*}{z}\overset{*}{\sigma}_k\partial_{\overset{*}{z}}) + \cdots,$$

$$\hat{B}_k = \frac{i}{2}(z\sigma_k\partial_z + \overset{*}{z}\overset{*}{\sigma}_k\partial_{\overset{*}{z}}) + \cdots, \quad z = (z^1\, z^2),\ \partial_z = (\partial/\partial z^1\ \partial/\partial z^2)^T; \tag{5.1}$$

$$\hat{S}^{\mathrm{R}}_k = -\frac{1}{2}(\chi\overset{*}{\sigma}_k\partial_\chi - \overset{*}{\chi}\sigma_k\partial_{\overset{*}{\chi}}) + \cdots,$$

$$\hat{B}^{\mathrm{R}}_k = -\frac{i}{2}(\chi\overset{*}{\sigma}_k\partial_\chi + \overset{*}{\chi}\sigma_k\partial_{\overset{*}{\chi}}) + \cdots, \quad \chi = (z^1\, \underline{z}^1),\ \partial_\chi = (\partial/\partial z^1\ \partial/\partial \underline{z}^1)^T; \tag{5.2}$$

Dots in the formulas replace analogous expressions obtained by the substitutions $z \to z' = (\underline{z}^1\, \underline{z}^2)$, $\chi \to \chi' = (z^2\, \underline{z}^2)$.

Since $\det Z = 1$, then any of $z_\alpha, \underline{z}_\alpha$ can be expressed in terms of the other three, for example, $\underline{z}_2 = (1 - z_2\underline{z}_1)/z_1$. The invariant measure on $\mathbb{R}^4 \times SL(2, C)$ has the form (Gel'fand *et al.*, 1966)

$$d\mu(x, \mathbf{z}) = (i/2)^3\, d^4x\, d^2z_1\, d^2z_2\, d^2\underline{z}_1\, |z_1|^{-2}. \tag{5.3}$$

The functions on the Poincaré group depend on 10 parameters, and correspondingly there are 10 commuting operators (two Casimir operators, four left generators, and four right generators).

Both the Poincaré group and the spin Lorentz subgroup have two Casimir operators:

$$\hat{p}^2 = \hat{p}_\mu \hat{p}^\mu, \quad \hat{W}^2 = \hat{W}_\mu \hat{W}^\mu, \quad \text{where } \hat{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\hat{p}_\nu \hat{J}_{\rho\sigma} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\hat{p}_\nu \hat{S}_{\rho\sigma}, \tag{5.4}$$

$$\frac{1}{2}\hat{S}_{\mu\nu}\hat{S}^{\mu\nu} = \frac{1}{2}\hat{S}^{\mathrm{R}}_{\mu\nu}\hat{S}^{\mu\nu}_{\mathrm{R}} = \hat{\mathbf{S}}^2 - \hat{\mathbf{B}}^2,$$

$$\frac{1}{16}\epsilon^{\mu\nu\rho\sigma}\hat{S}_{\mu\nu}\hat{S}_{\rho\sigma} = \frac{1}{16}\epsilon^{\mu\nu\rho\sigma}\hat{S}^{\mathrm{R}}_{\mu\nu}\hat{S}^{\mathrm{R}}_{\rho\sigma} = \hat{\mathbf{S}}\hat{\mathbf{B}}. \tag{5.5}$$

Let us consider a set of 10 commuting operators,

$$\hat{p}_\mu, \hat{W}^2, \hat{\mathbf{p}}\hat{\mathbf{S}}, \hat{\mathbf{S}}^2 - \hat{\mathbf{B}}^2, \hat{\mathbf{S}}\hat{\mathbf{B}}, \hat{S}^{\mathrm{R}}_3, \hat{B}^{\mathrm{R}}_3. \tag{5.6}$$

This set consists of operators of momenta, the Lubanski–Pauli operator $\hat{W}^2$, the operator $\hat{\mathbf{p}}\hat{\mathbf{J}} = \hat{\mathbf{p}}\hat{\mathbf{S}}$, which is proportional to the helicity, and four operators, which are functions of the right generators. These four operators commute with the left rotations and translations, and allow one to distinguish equivalent irreps in the decomposition of GRR. In the rest frame, $\hat{\mathbf{p}}\hat{\mathbf{S}} = 0$, and the complete set of commuting operators can be obtained from (5.6) with the help of the replacement of $\hat{\mathbf{p}}\hat{\mathbf{S}}$ by $\hat{S}_3$.

Functions $f(x, z)$ on the group $M(3, 1)$ are functions of four real variables $x^\mu$ and four complex variables $z_\alpha$, $\underline{z}_\alpha$ with the constraint $z_1\underline{z}_2 - z_2\underline{z}_1 = 1$.

The space of functions on the Poincaré group contains the subspace of analytic functions $f(x, z, \overset{*}{\underline{z}})$ of the elements of the Dirac $z$-spinor

$$Z_{\mathrm{D}} = (z^\alpha, \overset{*}{\underline{z}}_{\dot{\alpha}}). \tag{5.7}$$

Charge conjugation means the transition to the subspace of antianalytic functions (i.e., analytic functions of $\underline{z}^\alpha$, $\overset{*}{z}_{\dot{\alpha}}$).

According to (2.61), for the space inversion, we have $Z \xrightarrow{P} (Z^{-1})^\dagger$ or

$$\begin{pmatrix} z^1 & \underline{z}^1 \\ z^2 & \underline{z}^2 \end{pmatrix} \xrightarrow{P} \begin{pmatrix} -\overset{*}{\underline{z}}_{\dot{1}} & \overset{*}{z}_{\dot{1}} \\ -\overset{*}{\underline{z}}_{\dot{2}} & \overset{*}{z}_{\dot{2}} \end{pmatrix}, \tag{5.8}$$

This transformation reverses the sign of the boost operators $\hat{B}_k$. It is easy to see that, in contrast to charge conjugation, space inversion conserves the analyticity (or antianalyticity) of functions of $Z_{\mathrm{D}}$.

Similar to the three-dimensional case [see (4.5)], eigenfunctions of $\hat{S}^{\mathrm{R}}_3$ and $\hat{B}^{\mathrm{R}}_3$ differ only by a phase factor. Fixing eigenvalues of operators $\hat{S}^{\mathrm{R}}_3$ and $\hat{B}^{\mathrm{R}}_3$, one passes to the space of functions of $x^\mu$ and elements of the Majorana $z$-spinor

$$Z_{\mathrm{M}} = (z^\alpha, \overset{*}{z}_{\dot\alpha}), \tag{5.9}$$

that is, the space of functions of eight real, independent variables on the manifold

$$\mathbb{R}^4 \times \mathbb{C}^2, \quad d\mu = d^4x\, d^2z_1\, d^2z_2. \tag{5.10}$$

Thus, in this space we have eight commuting operators (two Casimir operators, four operators distinguishing states inside the irrep, and two operators distinguishing equivalent irreps). Notice that Kihlberg (1964) gives physical arguments for the necessity of using at least eight variables in order to describe spinning particles. The space reflection takes functions of $Z_{\mathrm{M}}$ to functions of $\underline{Z}_{\mathrm{M}} = (\underline{z}^\alpha, \overset{*}{\underline{z}}_{\dot\alpha})$; as mentioned above, $\underline{z}^\alpha$ and $z^\alpha$ have the same transformation rule. Charge conjugation leave the space of functions of $Z_{\mathrm{M}}$ invariant.

Below we will consider the massive case characterized by symmetry with respect to space reflection and therefore the space of the analytic functions of the Dirac $z$-spinor $Z_{\mathrm{D}}$, unless otherwise stipulated. In this space, the action of $M(3,1)$ is given by

$$\begin{aligned} T(g)f(x,z,\underline{z}) &= f(g^{-1}x, g^{-1}z, g^{-1}\underline{z}),\\ (g^{-1}x)^\mu &= (\Lambda^{-1})^\mu_\nu x^\nu, \quad (g^{-1}z)^\alpha = U^\alpha_\beta z^\beta,\\ (g^{-1}\underline{z})_{\dot\alpha} &= (U^{-1})^{\dot\beta}_{\dot\alpha}\, \overset{*}{\underline{z}}_{\dot\beta}. \end{aligned} \tag{5.11}$$

Spin projection operators have the form

$$\hat{S}_k = \frac{1}{2}(z\sigma_k\partial_z - \overset{*}{\underline{z}}\overset{*}{\sigma}_k\, \partial_{\overset{*}{\underline{z}}}), \quad \hat{B}_k = \frac{i}{2}(z\sigma_k\partial_z + \overset{*}{\underline{z}}\overset{*}{\sigma}_k\, \partial_{\overset{*}{\underline{z}}}). \tag{5.12}$$

One can compose the combinations $\hat{M}_k$, $\hat{\bar{M}}_k$

$$\begin{aligned} \hat{M}_k &= \frac{1}{2}(\hat{S}_k - i\hat{B}_k) = z\sigma_k\partial_z, \quad \hat{M}_+ = z^1\partial/\partial z^2, \quad \hat{M}_- = z^2\partial/\partial z^1,\\ \hat{\bar{M}}_k &= -\frac{1}{2}(\hat{S}_k + i\hat{B}_k) = \overset{*}{\underline{z}}\overset{*}{\sigma}_k\, \partial_{\overset{*}{\underline{z}}}, \quad \hat{\bar{M}}_+ = \overset{*}{\underline{z}}{}^{\dot 1}\, \partial/\partial\, \overset{*}{\underline{z}}{}^{\dot 2}, \quad \hat{\bar{M}}_- = \overset{*}{\underline{z}}{}^{\dot 2}\, \partial/\partial\, \overset{*}{\underline{z}}{}^{\dot 1}, \end{aligned} \tag{5.13}$$

such that $[\hat{M}_i, \hat{\bar{M}}_k] = 0$. For unitary representations of the Lorentz group, $\hat{S}_k^\dagger = \hat{S}_k$, $\hat{B}_k^\dagger = \hat{B}_k$, and these operators obey the relation $\hat{M}_k^\dagger = \hat{\bar{M}}_k$ (for finite-dimensional nonunitary irreps, $\hat{S}_k^\dagger = \hat{S}_k$, $\hat{B}_k^\dagger = -\hat{B}_k$, and $\hat{M}_k^\dagger = -\hat{\bar{M}}_k$). Introducing spin operators with spinor indices

$$\hat{M}_{\alpha\beta} = (\sigma_{\mu\nu})_{\alpha\beta}\hat{S}^{\mu\nu}, \quad \hat{\bar{M}}_{\dot\alpha\dot\beta} = (\bar\sigma_{\mu\nu})_{\dot\alpha\dot\beta}\hat{S}^{\mu\nu}, \tag{5.14}$$

where $\sigma_{\mu\nu}$ and $\bar\sigma_{\mu\nu}$ are defined in (B6), we obtain

$$\hat{S}^{\mu\nu} = -\frac{1}{2}\left[(\sigma^{\mu\nu})^{\alpha\beta}\hat{M}_{\alpha\beta} + (\bar\sigma^{\mu\nu})^{\dot\alpha\dot\beta}\hat{\bar{M}}_{\dot\alpha\dot\beta}\right], \tag{5.15}$$

$$\hat{M}_{\alpha\beta}\hat{M}^{\alpha\beta} = 2\hat{\mathbf{M}}^2, \quad \hat{\bar{M}}_{\dot{\alpha}\dot{\beta}}\hat{\bar{M}}^{\dot{\alpha}\dot{\beta}} = 2\hat{\bar{\mathbf{M}}}^2. \tag{5.16}$$

In the space of analytic functions of $z, \underline{\overset{*}{z}}$, we have

$$\hat{M}_{\alpha\beta} = \frac{1}{2}(z_\alpha\partial_\beta + z_\beta\partial_\alpha), \quad \hat{\bar{M}}_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}(\underline{\overset{*}{z}}_{\dot{\alpha}}\,\underline{\partial}_{\dot{\beta}} + \underline{\overset{*}{z}}_{\dot{\beta}}\,\underline{\partial}_{\dot{\alpha}}). \tag{5.17}$$

Taking into account that operators $\hat{M}_k$ and $\hat{\bar{M}}_k$ are subjected to commutation relations of $su(2)$ algebra, we obtain the spectra of the Casimir operators of the Lorentz subgroup:

$$\hat{\mathbf{S}}^2 - \hat{\mathbf{B}}^2 = 2(\hat{\mathbf{M}}^2 + \hat{\bar{\mathbf{M}}}^2) = 2j_1(j_1+1) + 2j_2(j_2+1) = -\frac{1}{2}(k^2 - \rho^2 - 4),$$
$$\hat{\mathbf{S}}\hat{\mathbf{B}} = -i(\hat{\mathbf{M}}^2 - \hat{\bar{\mathbf{M}}}^2) = -i(j_1(j_1+1) - j_2(j_2+1)) = k\rho, \tag{5.18}$$

where $\rho = -i(j_1 + j_2 + 1)$ and $k = j_1 - j_2$. Thus, irreps of the Lorentz group $SL(2, C)$ are labeled by the pair $(j_1, j_2)$. It is convenient to label unitary irreps by $[k, \rho]$, where irreps $[k, \rho]$ and $[-k, -\rho]$ are equivalent (Barut and Raczka, 1977; Gel'fand *et al.*, 1966).

Notice that the formulas (5.11)–(5.18) are also valid if, using the substitution $\underline{\overset{*}{z}}_{\dot{\alpha}} \to \overset{*}{z}_{\dot{\alpha}}$, we consider the functions of elements of the Majorana $z$-spinor $Z_{\mathrm{M}}$ instead of $Z_{\mathrm{D}}$.

The formulas of reduction on the compact $SU(2)$ subgroup have the form

$$T_{(j_1, j_2)} = \sum_{j=|j_1-j_2|}^{j_1+j_2} T_j, \quad T_{[k,\rho]} = \sum_{j=k}^{\infty} T_j \tag{5.19}$$

for finite-dimensional nonunitary irreps and infinite-dimensional unitary irreps of $SL(2, C)$, respectively (Gel'fand *et al.*, 1966). Analogous to the $2+1$ case, there are two types of Poincaré group representations describing the same spin $s$. These types correspond to finite-dimensional and infinite-dimensional unitary representations of the Lorentz group. In particular, one may choose (i) $s = j_{\max} = j_1 + j_2$ for *nonunitary finite-dimensional* irreps $(j_1, j_2)$ and (ii) $s = j_{\min} = j_0 = |j_1 - j_2|$ for *unitary infinite-dimensional* irreps $[j_0, p]$, where $j_{\max}$ and $j_{\min}$ are respectively the maximal and minimal $j$ in the decomposition (5.19) of an irrep of the Lorentz group over irreps $T_j$ of the compact $SU(2)$ subgroup. Below we will study only the case of finite-dimensional representations of the Lorentz group.

Consider the monomial basis

$$(z^1)^a(z^2)^b(\underline{\overset{*}{z}}_1)^c(\underline{\overset{*}{z}}_2)^d$$

in the space of functions $\phi(z, \underline{\overset{*}{z}})$. The values $j_1 = (a+b)/2$ and $j_2 = (c+d)/2$ are conserved under the action of the generators (5.13). Therefore, the space of the irrep $(j_1, j_2)$ is the space of homogeneous analytic functions depending on two pairs of complex variables of power $(2j_1, 2j_2)$. We denote these functions as $\phi_{j_1 j_2}(z, \overset{*}{\underline{z}})$.

For finite-dimensional nonunitary irreps of $SL(2, C)$, $a$, $b$, $c$, and $d$ are integer nonnegative, therefore $j_1$, $j_2$ are integer or half-integer nonnegative numbers. One can write functions $f_s(x, z, \overset{*}{\underline{z}})$, which are polynomial of the power $2s = 2j_1 + 2j_2$ in $z, \overset{*}{\underline{z}}$, in the form

$$f_s\big(x, z, \overset{*}{\underline{z}}\,\big) = \sum_{j_1+j_2=s} \sum_{m_1,m_2} \psi^{m_1 m_2}_{j_1 j_2}(x)\varphi^{m_1 m_2}_{j_1 j_2}(z, \overset{*}{\underline{z}}), \tag{5.20}$$

where the functions

$$\varphi^{m_1 m_2}_{j_1 j_2}(z, \overset{*}{\underline{z}}) = N^{1/2}(z^1)^{j_1+m_1}(z^2)^{j_1-m_1}(\overset{*}{\underline{z}}_{\dot 1})^{j_2+m_2}(\overset{*}{\underline{z}}_{\dot 2})^{j_2-m_2}, \tag{5.21}$$

$$N = (2s)!\,[(j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!]^{-1}, \tag{5.22}$$

form a basis of the irrep of the Lorentz group. This basis corresponds to the chiral representation (see Appendix B). On the other hand, one can write the decomposition of the same function in terms of the symmetric multispinors

$$\psi^{\dot\beta_1\cdots\dot\beta_{2j_2}}_{\alpha_1\cdots\alpha_{2j_1}}(x) = \psi^{\dot\beta_{(1}\cdots\dot\beta_{2j_2)}}_{\alpha_{(1}\cdots\alpha_{2j_1)}}(x)$$

as follows:

$$f_s\big(x, z, \overset{*}{\underline{z}}\,\big) = \sum_{j_1+j_2=s} f_{j_1 j_2}(x, z, \overset{*}{\underline{z}}),$$

$$f_{j_1 j_2}(x, z, \overset{*}{\underline{z}}) = \psi^{\dot\beta_1\cdots\dot\beta_{2j_2}}_{\alpha_1\cdots\alpha_{2j_1}}(x) z^{\alpha_1}\cdots z^{\alpha_{2j_1}}\overset{*}{\underline{z}}_{\dot\beta_1}\cdots\overset{*}{\underline{z}}_{\dot\beta_{2j_2}}. \tag{5.23}$$

Notice that similar generating functions summed over all $s$ have been used in Vasiliev (1992, 1996) to describe massless fields. Comparing decompositions (5.20) and (5.23), we obtain the relation

$$N^{1/2}\psi^{m_1 m_2}_{j_1 j_2}(x) = \psi^{\overbrace{\dot 1\ldots\dot 1}^{j_2+m_2}\,\overbrace{\dot 2\ldots\dot 2}^{j_2-m_2}}_{\underbrace{1\ldots 1}_{j_1+m_1}\,\underbrace{2\ldots 2}_{j_1-m_1}}(x). \tag{5.24}$$

Using the invariant tensor $\sigma^{\mu}_{\dot\alpha\alpha}$ and spinors $z^\alpha$, $\overset{*}{\underline{z}}_{\dot\alpha}$, $\partial_\alpha = \partial/\partial z^\alpha$, $\underline{\partial}^{\dot\alpha} = \partial/\partial\overset{*}{\underline{z}}_{\dot\alpha}$, it is possible to construct just four vectors:

$$\hat V^{\mu}_{12} = \frac{1}{2}\bar\sigma^{\mu\dot\alpha\alpha}\,\overset{*}{\underline{z}}_{\dot\alpha}\,\partial_\alpha, \quad \hat V^{\mu}_{21} = \frac{1}{2}\sigma^{\mu}_{\alpha\dot\alpha} z^\alpha\,\underline{\partial}^{\dot\alpha}, \tag{5.25}$$

$$\hat V^{\mu}_{11} = \frac{1}{2}\sigma^{\mu}_{\alpha\dot\alpha} z^\alpha\,\overset{*}{\underline{z}}^{\dot\alpha}, \qquad \hat V^{\mu}_{22} = \frac{1}{2}\sigma^{\mu}_{\alpha\dot\alpha}\partial^\alpha\,\underline{\partial}^{\dot\alpha}. \tag{5.26}$$

These operators are not functions of generators of $M(3, 1)$ and relate irreps with different $(j_1, j_2)$; as we will see below, they play an important role in the theory of RWE.

### 5.2. Relativistic Wave Equations Invariant Under the Proper Poincaré Group

Let us fix eigenvalues of the Casimir operators of the Poincaré group and of the Lorentz subgroup:

$$\hat{p}^2 f(x,\mathbf{z}) = m^2 f(x,\mathbf{z}), \tag{5.27}$$

$$\hat{W}^2 f(x,\mathbf{z}) = -s(s+1)m^2 f(x,\mathbf{z}), \tag{5.28}$$

$$\hat{\mathbf{M}}^2 f(x,\mathbf{z}) = j_1(j_1+1) f(x,\mathbf{z}), \tag{5.29}$$

$$\hat{\tilde{\mathbf{M}}}^2 f(x,\mathbf{z}) = j_2(j_2+1) f(x,\mathbf{z}), \tag{5.30}$$

The spectrum (5.28) of the operator $\hat{W}^2$ corresponds to the consideration of massive spin-$s$ particles and massless particles with discrete spin. [For tachyons and massless particles with continuous spin spectrum different from (5.28), see Barut and Raczka (1977) and Mackey (1968)]. As a consequence of the last two equations (recall that we consider the space of analytic functions of $z, \overset{*}{\underline{z}}$), we obtain that eigenvalues of the operators $\hat{S}_3^{\mathrm{R}}$ and $\hat{B}_3^{\mathrm{R}}$ belonging to the complete set (5.6) are also fixed,

$$\hat{S}_3^{\mathrm{R}} f(x,z,\overset{*}{\underline{z}}) = -(j_1+j_2) f(x,z,\overset{*}{\underline{z}}),$$

$$i\hat{B}_3^{\mathrm{R}} f(x,z,\overset{*}{\underline{z}}) = (j_1-j_2) f(x,z,\overset{*}{\underline{z}}). \tag{5.31}$$

Equations (5.27)–(5.30) define the reducible representation of the proper Poincaré group $M(3,1)$. This representation splits into two representations labeled by the sign of $p_0$ and are irreducible for $m \neq 0$.

Nonequivalent representations are distinguished by eigenvalues of the Casimir operators $\hat{p}^2$ and $\hat{W}^2$ and by the sign of $p_0$ [see also Barut and Raczka (1977), Mackey (1968), and Kim and Noz (1986)]. The case of zero eigenvalues of the operators $\hat{p}^2$ and $\hat{W}^2$ is an exception. This case corresponds to massless particles with discrete spin, and nonequivalent irreps are labeled by the helicity and by the sign of $p_0$. On the other hand, one can introduce a chirality as $\lambda = j_1 - j_2$ (or as the difference in the numbers of dotted and undotted indices). The explicit form of the chirality operator in the space of analytic functions of $z, \overset{*}{z}$ is given by [see (B4)]

$$\hat{\Gamma}^5 = \frac{1}{2}\left(z^\alpha \partial_\alpha - \overset{*}{\underline{z}}_{\dot{\alpha}}\, \underline{\partial}^{\dot{\alpha}}\right). \tag{5.32}$$

In the massless case, helicity is equal to chirality up to sign (Barut and Raczka, 1977). In the massive case, irreps of the proper Poincaré group, which are labeled by the same $m$, $s$, and sign $p_0$ but by different chiralities, are equivalent. Thus, for fixed mass $m$ and spin $s = j_1 + j_2$, the system (5.27)–(5.30) has $2s+1$ solutions differing by $\lambda = j_1 - j_2$.

Using (5.15), we can rewrite the Lubanski–Pauli vector (5.4) and the Casimir operator $\hat{W}^2$ in the form

$$\hat{W}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\hat{p}_\nu \hat{S}_{\rho\sigma} = \frac{1}{2}i\hat{p}_\nu\big[(\sigma^{\mu\nu})_{\alpha\beta}\hat{M}^{\alpha\beta} - (\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}\hat{\bar{M}}^{\dot{\alpha}\dot{\beta}}\big], \tag{5.33}$$

$$\hat{W}^2 = -\hat{p}^2(\hat{\mathbf{M}}^2 + \hat{\bar{\mathbf{M}}}^2) - \frac{1}{2}\hat{p}_\mu\hat{p}_\nu(\sigma^{\mu\rho})_{\alpha\beta}\big(\bar{\sigma}_\rho^\nu\big)_{\dot{\alpha}\dot{\beta}}\hat{M}^{\alpha\beta}\hat{\bar{M}}^{\dot{\alpha}\dot{\beta}}. \tag{5.34}$$

Taking into account the explicit form of the spin operators (5.17) and the symmetry of $(\sigma^{\mu\nu})_{\alpha\beta}$ and $(\bar{\sigma}^{\mu\nu})_{\dot{\alpha}\dot{\beta}}$ with respect to the permutation of spinor indices, we rewrite the last relation as

$$\hat{W}^2 = -\hat{p}^2(\hat{\mathbf{M}} + \hat{\bar{\mathbf{M}}}^2) - 2\hat{p}_\mu\hat{p}_\nu(\sigma^{\mu\rho})_{\alpha\beta}\big(\bar{\sigma}_\rho^\nu\big)_{\dot{\alpha}\dot{\beta}} z^\alpha\partial^\beta\, \underline{\overset{*}{z}}^{\dot{\alpha}}\, \underline{\partial}^{\dot{\beta}}.$$

Finally, using the identity

$$4(\sigma^{\mu\rho})_{\alpha\beta}\big(\bar{\sigma}_\rho^\nu\big)_{\dot{\alpha}\dot{\beta}} = -\eta^{\mu\nu}\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}} + \sigma^\mu_{\alpha\dot{\alpha}}\sigma^\nu_{\beta\dot{\beta}} + \sigma^\nu_{\alpha\dot{\alpha}}\sigma^\mu_{\beta\dot{\beta}}$$

and the condition of mass irreducibility (5.27), we obtain

$$\hat{W}^2 = -m^2(j_1 + j_2)(j_1 + j_2 + 1) + 4\hat{p}_\mu\hat{V}^\mu_{11}\hat{p}_\nu\hat{V}^\nu_{22}, \tag{5.35}$$

where the operators $\hat{V}^\mu_{11}$ and $\hat{V}^\mu_{22}$ are defined in (5.26). Therefore, for $s = j_1 + j_2$, the necessary and sufficient condition of spin irreducibility is

$$\hat{p}_\mu\hat{V}^\mu_{11}\hat{p}_\nu\hat{V}^\nu_{22} f(x, z, \underline{\overset{*}{z}}) = 0. \tag{5.36}$$

For the representations $(s\ 0)$ and $(0\ s)$, we have $\hat{V}^\mu_{22} f(x, z, \underline{\overset{*}{z}}) = 0$ and the condition (5.36) is fulfilled identically. In the general case, observing that in the momentum representation the action of the operator $\hat{V}^\mu_{11}\hat{p}_\mu$ reduces to multiplication by the number $p_\mu\sigma^\mu_{\alpha\dot{\alpha}}z^\alpha z^{\dot{\alpha}}$, we come to the *alternative conditions*

$$\hat{p}_\mu\hat{V}^\mu_{11} = 0, \tag{5.37}$$

$$\hat{p}_\nu\hat{V}^\nu_{22} f(x, z, \underline{\overset{*}{z}}) = 0. \tag{5.38}$$

The first condition connects the components of momentum $p_\mu$ and complex spin variables $q_\mu = \sigma_{\mu\alpha\dot{\alpha}}z^\alpha\underline{\overset{*}{z}}^{\dot{\alpha}}/2$, $q_\mu q^\mu = 0$. Thus, we have the space of functions of two four-vectors $p_\mu, q_\mu$, which are subject to the invariant constraints

$$p^2 = m^2, \quad p_\mu q^\mu = 0, \quad q^2 = 0. \tag{5.39}$$

According to (5.39), in the rest frame, we get $z^1\underline{\overset{*}{z}}^1 + z^2\underline{\overset{*}{z}}^2 = 0$. A similar approach to constructing wave functions describing elementary particles was suggested by Wigner (1963), who restricted discussion to particles of integer spin and real $q_\mu$ with constraints $p^2 = m^2$, $p_\mu q^\mu = 0$, $q^2 = -1$. Different generalizations of his approach (Wigner, 1963) have been considered (Biedenharn *et al.*, 1988; Hasiewicz and Siemion, 1992; Kim and Wigner, 1987; Kuzenko *et al.*, 1995; Lyakhovich *et al.*, 1996).

The second condition (5.38) does not affect spin variables and can be written in terms of $\psi(x)$. For fixed $j_1, j_2$, using the decomposition (5.23) of $f(x, z, \overset{*}{\underline{z}})$ in terms of multispinors and also the relation $\partial_\alpha \psi_{\alpha_1\alpha_2\ldots\alpha_k} z^{\alpha(1} z^{\alpha_2} \ldots z^{\alpha_k)} = \delta_\alpha^{\alpha_1} \psi_{\alpha_1\alpha_2\ldots\alpha_k} z^{\alpha(2} \ldots z^{\alpha_k)}$, one can rewrite the system

$$(\hat{p}^2 - m^2) f_{j_1 j_2}(x, z) = 0, \quad \hat{p}_\mu \sigma^\mu_{\alpha\dot{\alpha}} \partial^\alpha \underline{\partial}^{\dot{\alpha}} f_{j_1 j_2}(x, z) = 0 \tag{5.40}$$

in the form

$$(\hat{p}^2 - m^2)\psi_{\alpha_1\ldots\alpha_k\dot{\alpha}_1\ldots\dot{\alpha}_l}(x) = 0, \tag{5.41}$$

$$\partial^{\dot{\alpha}\alpha}\psi_{\alpha\alpha_1\ldots\alpha_{k-1}\dot{\alpha}\dot{\alpha}_1\ldots\dot{\alpha}_{l-1}}(x) = 0, \tag{5.42}$$

where $\partial^{\dot{\alpha}\alpha} = \partial_\mu \bar{\sigma}^{\mu\dot{\alpha}\alpha}$, $k = 2j_1$, $l = 2j_2$. These equations describe a particle with unique mass $m$ and spin $s = j_1 + j_2$. The subsidiary condition (5.42) is necessary to exclude components corresponding to other possible spins $s$, $|j_1 - j_2| \le s < j_1 + j_2$; see (5.19).

On the other hand, in order to describe spin $s$, one can use representations $(j_1 j_2)$, $j_1 + j_2 \ne s$. In this case, according to (5.35), the condition (5.42) should be replaced by the new subsidiary condition

$$\partial_{\beta\dot{\beta}}\partial^{\dot{\alpha}\alpha}\psi_{\alpha\alpha_1\ldots\alpha_{k-1}\dot{\alpha}\dot{\alpha}_1\ldots\dot{\alpha}_{l-1}}(x) = -m^2[(j_1 + j_2)(j_1 + j_2 + 1) \\ -s(s+1)]\psi_{\beta\alpha_1\ldots\alpha_{k-1}\dot{\beta}\dot{\alpha}_1\ldots\dot{\alpha}_{l-1}}(x). \tag{5.43}$$

Note that an approach using this general subsidiary condition was not considered earlier.

Passing on to vector indices, one can see that for integer spins and irreps $(\frac{s}{2}\,\frac{s}{2})$, Eqs. (5.41) and (5.42) take the form

$$(\hat{p}^2 - m^2)\Phi_{\mu_1\mu_2\ldots\mu_s}(x) = 0, \quad \partial^\mu \Phi_{\mu\mu_2\ldots\mu_s}(x) = 0, \quad \Phi^\mu_{\mu\mu_2\ldots\mu_s}(x) = 0, \tag{5.44}$$

where

$$\Phi_{\mu_1\mu_2\ldots\mu_s}(x) = (-1)^s 2^{-s} \bar{\sigma}^{\dot{\alpha}_1\alpha_1}_{\mu_1} \cdots \bar{\sigma}^{\alpha_s\alpha_s}_{\mu_s} \psi_{\alpha_1\ldots\alpha_s\dot{\alpha}_1\ldots\dot{\alpha}_s}(x).$$

Equations (5.44), known also as the massive tensor field equations or Fierz–Pauli equations, are used most often to describe integer spins.

For half-integer spins and irreps $(\frac{2s\pm1}{4}\,\frac{2s\mp1}{4})$, after passage to vector indices, the subsidiary conditions (5.42) take the form

$$\partial^\mu \Psi_{\mu\mu_2\ldots\mu_n\alpha}(x) = 0, \quad \bar{\sigma}^{\mu\dot{\alpha}\alpha}\Psi_{\mu\mu_2\ldots\mu_n\alpha}(x) = 0, \quad \Psi^\mu_{\mu\mu_2\ldots\mu_n\alpha}(x) = 0,$$

$$\partial^\mu \Psi_{\mu\mu_2\ldots\mu_n\dot{\alpha}}(x) = 0, \quad \sigma^\mu_{\alpha\dot{\alpha}}\Psi^{\dot{\alpha}}_{\mu\mu_2\ldots\mu_n}(x) = 0, \quad \Psi^\mu_{\mu\mu_2\ldots\mu_n\dot{\alpha}}(x) = 0, \tag{5.45}$$

where $n = (2s - 1)/2$.

## 5.3. Relativistic Wave Equations Invariant Under the Improper Poincaré Group

The improper Poincaré group includes continuous transformations of the proper group and space reflection operator (parity operator) $\hat{I}_{\text{P}}$. According to (2.61) and (5.8), this operator obeys the condition $\hat{I}_{\text{P}}^2 = \hat{1}$ and the commutation relations

$$[\hat{I}_{\text{P}}, \hat{p}_0] = [\hat{I}_{\text{P}}, \hat{p}^2] = [\hat{I}_{\text{P}}, \hat{W}^2] = [\hat{I}_{\text{P}}, \hat{S}_k] = [\hat{I}_{\text{P}}, \hat{S}_k^{\text{R}}] = 0, \tag{5.46}$$

$$[\hat{I}_{\text{P}}, \hat{p}_k]_+ = [\hat{I}_{\text{P}}, \hat{B}_k]_+ = [\hat{I}_{\text{P}}, \hat{B}_k^{\text{R}}]_+ = 0. \tag{5.47}$$

States with definite total parity are defined as eigenfunctions of the operator $\hat{I}_{\text{P}}$:

$$\hat{I}_{\text{P}} f(x, \mathbf{z}) = \pm f(x, \mathbf{z}). \tag{5.48}$$

For $m > 0$, irreps of the improper Poincaré group are labeled by an orbit defining the mass $m$ and the sign of $p_0$ and by the irrep of the little group $O(3)$ defining spin $s$ and intrinsic parity (Barut and Raczka, 1977; Mackey, 1968; Tung, 1985). In the rest frame, the intrinsic parity coincides with the total.

The Casimir operators of the Lorentz group do not commute with the parity operator, $[\hat{I}_{\text{P}}, \hat{\mathbf{M}}^2] = \hat{\breve{\mathbf{M}}}^2$, $[\hat{I}_{\text{P}}, \hat{\breve{\mathbf{M}}}^2] = \hat{\mathbf{M}}^2$, and parity transformation combine two equivalent irreps labeled by Lorentz indices $(j_1,\ j_2)$ and $(j_2,\ j_1)$ (by chiralities $\pm\lambda$) of the proper Poincaré group into one representation of the improper group. The latter representation is reductible and splits into two irreps differing in intrinsic parity $\eta = \pm 1$. Thus, we cannot use the operators $\hat{\mathbf{M}}^2, \hat{\breve{\mathbf{M}}}^2$ to select invariant subspaces, and instead of the set of eight commuting operators

$$\hat{p}_\mu,\ \hat{W}^2,\ \hat{\mathbf{p}}\hat{\mathbf{S}},\ \hat{\mathbf{M}}^2,\ \hat{\breve{\mathbf{M}}}^2 \tag{5.49}$$

used above in order to construct the system (5.27)–(5.30), we should consider another set. Notice that parity operator $\hat{I}_{\text{P}}$ cannot be used directly for identification of invariant subspaces since, according to (5.47), it does not commute with translations and boosts.

The simplest possibility is to consider the system

$$\hat{p}^2 f(x, z, \overset{*}{\underline{z}}) = m^2 f(x, z, \overset{*}{\underline{z}}), \tag{5.50}$$

$$\hat{W}^2 f(x, z, \overset{*}{\underline{z}}) = -s(s+1)m^2 f(x, z, \overset{*}{\underline{z}}), \tag{5.51}$$

$$\hat{S}_3^{\text{R}} f(x, z, \overset{*}{\underline{z}}) = -sf(x, z, \overset{*}{\underline{z}}). \tag{5.52}$$

The last equation fixes the power $2s = 2(j_1 + j_2)$ of the polynomial in $z, \overset{*}{z}$; see (5.31). The first two equations are the conditions of mass and spin irreducibility. Therefore, the system describes fixed mass and spin, but the Poincaré group representation defined by this system is reducible. This representation decomposes into $2(2s+1)$ irreps differing in the chirality $\lambda = -s, \ldots, s$ and sign of $p_0$.

Supplementing the system (5.50)–(5.52) by the equation

$$i\hat{B}_3^{\mathrm{R}} f(x,z,\overset{*}{\underline{z}}) = \pm(j_1 - j_2) f(x,z,\overset{*}{\underline{z}}), \tag{5.53}$$

which change the sign under space reflection, it is possible to extract components corresponding to the representation $(j_1, j_2) \oplus (j_2, j_1)$. If we consider only the components labeled by $(j_1, j_2)$ and $(j_2, j_1)$, then for $j_1 \neq j_2$, the mass and spin irreducibility conditions (5.50) and (5.51) leave $4(2s+1)$ independent components corresponding to the direct sum of four improper Poincaré group irreps differing in the signs of the energy $p_0$ and the intrinsic parity $\eta$. But states with definite intrinsic parity arise in such an approach only as linear combinations of the solutions of *two* systems (5.50)–(5.53) with different sign in the last equation (i.e., solutions with fixed chirality).

Let us investigate the possibility of constructing a system of equations that remains invariant under space reflection and has solutions with definite intrinsic parity. For this purpose, it is necessary to consider equations that combine equivalent irreps of the proper Poincaré group labeled by different chiralities $\lambda = j_1 - j_2$. In the other words, *it is necessary to consider supplementary operators that define some extension of the Lorentz group.* These operators, replacing $\hat{\mathbf{M}}^2$ and $\hat{\tilde{\mathbf{M}}}^2$ in the set (5.49), must commute with all the left generators of the proper Poincaré group and with parity operator $\hat{I}_P$. *We suppose that one of these commuting operators is linear in* $\hat{p}$.

A general form of the invariant equations linear in $\hat{p}$ is

$$\hat{p}_\mu \hat{V}^\mu f(x,\mathbf{z}) = \varkappa f(x,\mathbf{z}), \tag{5.54}$$

where $\hat{V}^\mu$ transforms as a four-vector function of $\mathbf{z}$ and $\partial/\partial\mathbf{z}$.

The above vector operators $V_{ik}^\mu$ of (5.25), (5.26) connect irreps with different $(j_1, j_2)$. Operators $\hat{V}_{12}^\mu$, $\hat{V}_{21}^\mu$ conserve $j_1 + j_2$, and operators $\hat{V}_{11}^\mu$, $\hat{V}_{22}^\mu$ conserve $j_1 - j_2$. One can consider any of the relations connecting two scalar functions

$$\hat{p}_\mu \hat{V}_{12}^\mu f_{j_1,j_2}(x,z,\overset{*}{\underline{z}}) = \varkappa_{12} f_{j_1-\frac{1}{2},j_2+\frac{1}{2}}(x,z,\overset{*}{\underline{z}}),$$

$$\hat{p}_\mu \hat{V}_{21}^\mu f_{j_1,j_2}(x,z,\overset{*}{\underline{z}}) = \varkappa_{21} f_{j_1+\frac{1}{2},j_2-\frac{1}{2}}(x,z,\overset{*}{\underline{z}}), \tag{5.55}$$

$$\hat{p}_\mu \hat{V}_{11}^\mu f_{j_1,j_2}(x,z,\overset{*}{\underline{z}}) = \varkappa_{11} f_{j_1+\frac{1}{2},j_2+\frac{1}{2}}(x,z,\overset{*}{\underline{z}}),$$

$$\hat{p}_\mu \hat{V}_{22}^\mu f_{j_1,j_2}(x,z,\overset{*}{\underline{z}}) = \varkappa_{22} f_{j_1-\frac{1}{2},j_2-\frac{1}{2}}(x,z,\overset{*}{\underline{z}}), \tag{5.56}$$

as an RWE. Thus, the operator $\hat{V}^\mu$ in (5.54) is a linear combination of $\hat{V}_{ik}^\mu$.

Let us consider finite-component equations invariant with respect to space reflection. This means:

1. The operator $\hat{p}_\mu \hat{V}^\mu$ is invariant under space reflection.
2. The equation has solutions $f(x,z,\overset{*}{z}) = \sum \psi_n(x)\phi_n(z,\overset{*}{\underline{z}})$, where functions $\phi_n(z)$ carry a representation containing a finite number of irreps $(j_1, j_2)$.

It is easy to see that for $\mathcal{X}_{11} \neq 0$, the operator $\hat{V}^{\mu}_{11}$ cannot be contained in $\hat{V}^{\mu}$. In this case, for $\varkappa_{22} \neq 0$, one can separate from the system of equations for functions $f_{j_1, j_2}(x, z, \overset{*}{\underline{z}})$, $f(x, z, \overset{*}{\underline{z}}) = \sum f_{j_1, j_2}(x, z, \overset{*}{\underline{z}})$ the independent equation for the function characterized by maximal $j_1 + j_2$, which does not contain $\hat{V}^{\mu}_{22}$. (It is not necessary to use operators $\hat{V}^{\mu}_{11}$ and $\hat{V}^{\mu}_{22}$ since these operators leave $j_1 - j_2$ invariable and cannot connect irreps with different $\lambda$.)

Relating to operators $\hat{V}^{\mu}_{12}$ and $\hat{V}^{\mu}_{21}$, one can see that only the combination $\hat{p}_{\mu}\hat{\Gamma}^{\mu}$,

$$\hat{\Gamma}^{\mu} = \hat{V}^{\mu}_{12} + \hat{V}^{\mu}_{21} = \frac{1}{2}\left(\bar{\sigma}^{\mu\dot{\alpha}\alpha}\overset{*}{\underline{z}}_{\dot{\alpha}}\partial_{\alpha} + \sigma^{\mu}_{\alpha\dot{\alpha}} z^{\alpha}\underline{\partial}^{\dot{\alpha}}\right), \tag{5.57}$$

is invariant under space reflections. Operators $\hat{\Gamma}^{\mu}$ connect the representation $(j_1, j_2)$ with $(j_1 + 1\ j_2 - 1)$ and $(j_1 - 1\ j_2 + 1)$ and conserve $j_1 + j_2$. These operators obey the commutation relations

$$[\hat{S}^{\lambda\mu}, \hat{\Gamma}^{\nu}] = i(\eta^{\mu\nu}\hat{\Gamma}^{\lambda} - \eta^{\lambda\nu}\hat{\Gamma}^{\mu}), \tag{5.58}$$

$$[\hat{\Gamma}^{\mu}, \hat{\Gamma}^{\nu}] = -i\hat{S}^{\mu\nu}, \tag{5.59}$$

which coincide with the commutation relations of the matrices $\gamma^{\mu}/2$. An explicit calculation shows that $\hat{\Gamma}_{\mu}\hat{\Gamma}^{\mu}$ depends on the irrep of the Lorentz subgroup,

$$\hat{\Gamma}_{\mu}\hat{\Gamma}^{\mu} = 2j_1 + 2j_2 + 4j_1 j_2. \tag{5.60}$$

Supplementing the generators of the Lorentz group by the four operators

$$\hat{S}^{4\mu} = \hat{\Gamma}^{\mu}, \quad \hat{S}^{ab} = -\hat{S}^{ba}, \tag{5.61}$$

we obtain

$$[\hat{S}^{ab}, \hat{S}^{cd}] = i(\eta^{bc}\hat{S}^{ad} - \eta^{ac}\hat{S}^{bd} - \eta^{bd}\hat{S}^{ac} + \eta^{ad}\hat{S}^{bc}), \quad \eta^{44} = \eta^{00} = 1. \tag{5.62}$$

Thus, the operators $\hat{S}^{ab}$, $a, b = 0, 1, 2, 3, 4$, obey the commutation relations of the generators of the $3 + 2$ de Sitter group $SO(3, 2) \sim Sp(4, R)$. This group has two fundamental irreps, namely the four-dimensional spinor irrep $T_{[10]}$ [by matrices $Sp(4, R)$] and the five-dimensional vector irrep $T_{[01]}$ [by matrices $SO(3, 2)$].

Using (5.5), (5.18), and (5.60), we obtain for the second-order Casimir operator of the group $Sp(4, R)$

$$\hat{S}_{ab}\hat{S}^{ab} f(x, z, \overset{*}{\underline{z}}) = 4S(S + 2) f(x, z, \overset{*}{\underline{z}}), \quad S = j_1 + j_2.$$

Thus, we deal with symmetric representations of $Sp(4, R)$, which we denote as $T_{[2S0]}$ (see Appendix B). These irreps can be obtained as a symmetric term in the decomposition of the direct product $(\otimes T_{[10]})^{2S}$. Irreps $T_{[2S0]}$ characterized by dimensionality $(2S + 3)!/[6(2S)!]$ combine all finite-dimensional irreps of the Lorentz group with $j_1 + j_2 = S$.

However, it is obvious that the equation

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x, z, \overset{*}{\underline{z}}) = \varkappa f(x, z, \overset{*}{\underline{z}}) \tag{5.63}$$

by itself does not fix spin $s$ and mass $m$, defined by (5.27) and (5.28), or the power $j_1 + j_2$ of the $f(x, z, \overset{*}{\underline{z}})$ in $\underline{z}, \overset{*}{\underline{z}}$. In the rest frame, it is easy to see that even for fixed $S = j_1 + j_2$, this equation fixes only the product $ms = \varkappa$, $s \leq S$.

Let us consider the set of eight commuting operators

$$\hat{p}_\mu,\ \hat{W}^2,\ \hat{\mathbf{p}}\hat{\mathbf{S}} \text{ (or } \hat{S}_3 \text{ in the rest frame)},\ \hat{p}_\mu \hat{\Gamma}^\mu,\ \hat{S}_{ab}\hat{S}^{ab} \tag{5.64}$$

acting in the space of functions of eight variables $x^\mu, \underline{z}^\alpha, \overset{*}{z}_{\dot{\alpha}}$. In comparison with the set (5.49), we have replaced two right operators $\hat{\underline{\mathbf{M}}}^2$, $\hat{\mathbf{M}}^2$ by operators $\hat{p}_\mu \Gamma^\mu$, $\hat{S}_{ab}\hat{S}^{ab}$ invariant under parity transformation. Notice that instead of $\hat{S}_{ab}\hat{S}^{ab}$, one can use the operator $\hat{S}_3^{\mathrm{R}}$ with eigenvalues equal to the minus power of the polynomial in $z, \overset{*}{\underline{z}}$, see (5.52).

Invariant subspaces are labeled by eigenvalues of operators

$$\hat{p}^2 f(x, z, \overset{*}{\underline{z}}) = m^2 f(x, z, \overset{*}{\underline{z}}), \tag{5.65}$$

$$\hat{W}^2 f(x, z, \overset{*}{\underline{z}}) = -m^2 s(s+1) f(x, z, \overset{*}{\underline{z}}), \tag{5.66}$$

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x, z, \overset{*}{\underline{z}}) = m\tilde{s} f(x, z, \overset{*}{\underline{z}}), \tag{5.67}$$

$$\hat{S}_{ab}\hat{S}^{ab} f(x, z, \overset{*}{\underline{z}}) = 4S(S+2) f(x, z, \overset{*}{\underline{z}}). \tag{5.68}$$

Unlike Eqs. (5.29) and (5.30), which fix $j_1$ and $j_2$ separately, the last equation of the system fixes the irrep $T_{[2S0]}$ of the $3+2$ de Sitter group and therefore the power $2S = 2j_1 + 2j_2$ of the polynomial in $z, \overset{*}{\underline{z}}$. Irreps of the Poincaré group characterized by spin $s \leq S$ can be realized in the space of these polynomials.

In the rest frame,

$$\hat{p}_0^2 f(x, z, \overset{*}{\underline{z}}) = m^2 f(x, z, \overset{*}{\underline{z}}),$$

$$\hat{p}_0 \hat{\Gamma}^0 f(x, z, \overset{*}{\underline{z}}) = m\tilde{s} f(x, z, \overset{*}{\underline{z}}), \quad \hat{\Gamma}^0 = \frac{1}{2}\left(\sigma^{0\dot{\alpha}\alpha}\, \overset{*}{\underline{z}}_{\dot{\alpha}} \partial_\alpha + \sigma^0_{\alpha\dot{\alpha}} z^\alpha \underline{\partial}^{\dot{\alpha}}\right). \tag{5.69}$$

According to the first equation, $p_0 = \pm m$, and correspondingly $\tilde{s}$ is a product of eigenvalue of operator $\hat{\Gamma}^0$ and sign $p_0$. For $p_0 = m$, any function characterized by $n_1 - n_2 = 2s$ is the solution of Eq. (5.69), where $n_1$ is the power of homogeneity in the variables $(z^1 + \overset{*}{\underline{z}}_{\dot{1}}), (z^2 + \overset{*}{\underline{z}}_{\dot{2}})$, and $n_2$ is the power of homogeneity in the variables $(z^1 - \overset{*}{\underline{z}}_{\dot{1}}), (z^2 - \overset{*}{\underline{z}}_{\dot{2}})$. Therefore, for $p_0 = -m$, any function characterized by $n_1 - n_2 = -2s$ is the solution of Eq. (5.69).

Let us show that the relation

$$|\tilde{s}| \leq s \leq S \tag{5.70}$$

holds. The variables $z^\alpha$ and $z_{\dot\alpha}$ have the same transformation rule under space rotations. Thus, the pairs $(z^1 + \underline{\overset{*}{z}}_{\dot 1})$, $(z^2 + \underline{\overset{*}{z}}_{\dot 2})$ and $(z^1 - \underline{\overset{*}{z}}_{\dot 1})$, $(z^2 - \underline{\overset{*}{z}}_{\dot 2})$ are spinors of rotation group, but are characterized by opposite parity. The polynomials of power $2j'$ in the first pair of variables or $2j''$ in the second pair of variables transform under $T_{j'}$ or $T_{j''}$ of the rotation group. At fixed $j'$ and $j''$, the relation $\tilde{s} = (j' - j'')\,\mathrm{sign}\,p_0$ holds, and the space of polynomials of the power $2S = 2j' + 2j''$ corresponds to the direct product of the representations $T_{j'}$ and $T_{j''}$. This direct product decomposes into a sum of irreps, labeled by $s = |j' - j''|, \ldots, j' + j''$, and therefore spin $s$ runs from $|\tilde{s}|$ up to $S$.

In particular, for $|\tilde{s}| = S$, the spin irreducibility condition (5.66) is a consequence of other equations of the system, and the spin is equal to one half of the polynomial power. Below we restrict our consideration to this case, which allows us to describe the spin $s$ by means of the irrep of the $3+2$ de Sitter group with minimal possible dimensionality. Correspondingly, for $\tilde{s} = S$, we will consider the system

$$\hat{p}^2 f(x, \mathbf{z}) = m^2 f(x, \mathbf{z}), \tag{5.71}$$

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x, \mathbf{z}) = msf(x, \mathbf{z}), \tag{5.72}$$

$$\hat{S}_{ab}\hat{S}^{ab} f(x, \mathbf{z}) = 4s(s+2) f(x, \mathbf{z}). \tag{5.73}$$

In the rest frame, the general solution in the set of the polynomial of power $2s$ in $z, \underline{\overset{*}{z}}$ has the form

$$f_{m,s}(x, \mathbf{z}) = \sum_{s_3=-s}^{s} C_{s_3} e^{imx^0} \left(z^1 + \underline{\overset{*}{z}}_{\dot 1}\right)^{s+s_3} \left(z^2 + \underline{\overset{*}{z}}_{\dot 2}\right)^{s-s_3}$$
$$+ C'_{s_3} e^{-imx^0} \left(z^1 - \underline{\overset{*}{z}}_{\dot 1}\right)^{s+s_3} \left(z^2 - \underline{\overset{*}{z}}_{\dot 2}\right)^{s-s_3}, \tag{5.74}$$

where $s_3$ is the spin projection,

$$\hat{S}_3 f(x, \mathbf{z}) = s_3 f(x, \mathbf{z}), \quad \hat{S}_3 = \frac{1}{2}\left(z^1 \partial_1 + \underline{\overset{*}{z}}_{\dot 1} \underline{\partial}^{\dot 1} - z^2 \partial_2 - \underline{\overset{*}{z}}_{\dot 2} \underline{\partial}^{\dot 2}\right). \tag{5.75}$$

Thus, for unique mass $m$ and spin $s$, there are $2s+1$ independent positive-frequency solutions and $2s+1$ independent negative-frequency solutions belonging to two irreps of the improper Poincaré group. In the case $\tilde{s} = -S$, which corresponds to the change of sign in Eq. (5.72), the general solution is obtained from (5.74) by the substitution $(z^\alpha + \underline{\overset{*}{z}}_{\dot\alpha}) \leftrightarrow (z^\alpha - \underline{\overset{*}{z}}_{\dot\alpha})$. It follows from (5.74) that for half-integer spins the sign of $\tilde{s}$ is the product of sign $p_0$ and the intrinsic parity.[9]

[9] According to (5.74), in the rest frame for half-integer spin, positive-frequency and negative-frequency states are characterized by opposite parity. One can show (Ahluwalia et al., 1993; Gaioli and Alvarez, 1995; Gavrilov and Gitman, 2000; Ryder, 1988) that for fixed mass $m$ and representation $(\frac{1}{2}\ 0) \oplus (0\ \frac{1}{2})$ of the Lorentz group, this condition is sufficient to derive the Dirac equation.

Only the four-dimensional irrep of the 3 + 2 de Sitter group corresponding to spin 1/2 remains irreducible under the reduction on the improper Lorentz group. For spin 1, the 10-dimensional irrep splits into $6 + 4$ (antisymmetric tensor and four-vector), and for spin 3/2, the 20-dimensional irrep splits into $8 + 12$, as so on.

Consider plane wave solutions corresponding to a particle moving along $x^3$. They can be obtained from the solutions in the rest frame (5.74) by means of the Lorentz transformation

$$P = UP_0U^\dagger, \quad \text{where } P_0 = \pm\mathrm{diag}\{m, m\}, \quad U = \mathrm{diag}\{e^{-a}, e^a\} \in SL(2, C),$$

where the sign corresponds to the sign of $p_0$,

$$p_\mu = k_\mu \,\mathrm{sign}\, p_0, \quad k_0 = m\cosh 2a, \quad k_3 = m\sinh 2a, \quad e^{\pm a} = \sqrt{(k_0 \pm k_3)/m}. \tag{5.76}$$

Thus, it follows that

$$f'_{m,s,s_3}(x, \mathbf{z}) = C_1 e^{ik_0x^0 + k_3x^3}\left(z^1e^a + \overset{*}{\underline{z}}_{\dot 1}\, e^{-a}\right)^{s+s_3}\left(z^2e^{-a} + \overset{*}{\underline{z}}_{\dot 2}\, e^{a}\right)^{s-s_3}$$
$$+ C_2 e^{ik_0x^0 - k_3x^3}\left(z^1e^a - \overset{*}{\underline{z}}_{\dot 1}\, e^{-a}\right)^{s+s_3}\left(z^2e^{a} - \overset{*}{\underline{z}}_{\dot 2}\, e^{-a}\right)^{s-s_3}. \tag{5.77}$$

In the ultrarelativistic case, for positive $a$, it is convenient to rewrite (5.77) in the form

$$f_{m,s,s_3}(x, \mathbf{z}) = \left(\frac{k_0 + k_3}{m}\right)^s \left\{\left[C_1 e^{ik_0x^0 + k_0x^3} + C_2(-1)^{s-s_3}e^{-ik_0x^0 - k_0x^3}\right]\right.$$
$$\left. \times\, (z^1)^{s+s_3}(\overset{*}{\underline{z}}_{\dot 2})^{s-s_3} + O\left(\frac{k_0 - k_3}{k_0 + k_3}\right)^{\frac{1}{2}}\right\} \tag{5.78}$$

The main term in (5.78) corresponds to functions carrying irrep $(\frac{s+\lambda}{2}\ \frac{s-\lambda}{2})$, $\lambda = s_3$, of the Lorentz group. The contributions of other irreps, $(\frac{s+\lambda'}{2}\ \frac{s-\lambda'}{2})$, are damped by a factor $(\frac{k^0-k_3}{k^0+k_3})^{|\lambda-\lambda'|}$. Passing to the limit $a \to +\infty$, (or $m \to 0$), we obtain the states with certain chirality $\lambda = j_1 - j_2 = s_3$ (for $a \to -\infty$, with chirality $\lambda = j_1 - j_2 = -s_3$, respectively). In particular, in the limit, the states characterized by helicity $s_3 = \pm s$ correspond to the representation $(s\,0) \oplus (0\,s)$ of the Lorentz group.

Taking into account that operators $\hat V^\mu_{21}(\hat V^\mu_{12})$ lower (raise) chirality $\lambda$ by 1 and the decomposition

$$f_s(x, z, \overset{*}{\underline{z}}) = \sum_{\lambda=-s}^{s} f_{j_1j_2}(x, z, \overset{*}{\underline{z}}), \quad \text{where } s = j_1 + j_2, \quad \lambda = j_1 - j_2, \tag{5.79}$$

one can write Eq. (5.72) in chiral representation in the form

$$\begin{pmatrix} \hat{p}_\mu \hat{V}^\mu_{21} f_{s-\frac{1}{2},\frac{1}{2}} \\ \hat{p}_\mu \hat{V}^\mu_{12} f_{s,0} + \hat{p}_\mu \hat{V}^\mu_{21} f_{s-1,1} \\ \cdots \\ \hat{p}_\mu \hat{V}^\mu_{12} f_{\frac{1}{2},s-\frac{1}{2}} \end{pmatrix} = ms \begin{pmatrix} f_{s,0} \\ f_{s-\frac{1}{2},\frac{1}{2}} \\ \cdots \\ f_{0,s} \end{pmatrix}, \tag{5.80}$$

For $m \neq 0$, this equation binds $1 + [s]$ irreps of the improper Lorentz group and allows one to express components corresponding to the irrep $(s\,0)$ in terms of components corresponding to the irrep $(s - \frac{1}{2}\;\frac{1}{2})$ and so on. This, in turn, for $s = 1, 3/2, 2$, allows one to pass from the first-order equations for the reducible representation to second-order equations for the irrep of the improper Poincaré group. For example, for $s = 1$, excluding $f_{1,0}$ and $f_{0,1}$, we obtain

$$m^2 f_{\frac{1}{2}\frac{1}{2}}(x, \mathbf{z}) = \left[\hat{p}_\mu \hat{V}^\mu_{12}, \hat{p}_\nu \hat{V}^\nu_{21}\right]_+ f_{\frac{1}{2}\frac{1}{2}}(x, \mathbf{z}). \tag{5.81}$$

In the general case, one also can to pass from the system of first-order equations (5.80) on the reducible representation to higher order equations for the irrep, for example, to the equations of order $1 + [s]$ on the components transforming under irreps $(\frac{s}{2}\;\frac{s}{2})$ or $(\frac{2s+1}{4}\;\frac{2s-1}{4}) \oplus (\frac{2s-1}{4}\;\frac{2s+1}{4})$ for the cases of integer or half-integer spin, respectively.

Let us consider some particular cases.

For $s = j_1 + j_2 = 1/2$, we have

$$f_{\frac{1}{2}}(x, \mathbf{z}) = \chi_\alpha(x) z^\alpha + \overset{*}{\psi}{}^{\dot\alpha}(x)\, \underline{\overset{*}{z}}_{\dot\alpha} = Z_{\rm D} \Psi_{\rm D}(x), \quad \Psi_{\rm D}(x) = \begin{pmatrix} \chi_\alpha(x) \\ \overset{*}{\psi}{}^{\dot\alpha}\,(x) \end{pmatrix}, \tag{5.82}$$

where $Z_{\rm D}$ is given by formula (5.7). If we substitute (5.82) into Eq. (5.72) and compare the coefficients at $z^\alpha$ and at $\underline{\overset{*}{z}}_{\dot\alpha}$ in the left and right sides, we obtain the Dirac equation

$$\hat{p}_\mu \gamma^\mu \Psi_{\rm D}(x) = m \Psi_{\rm D}(x), \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{5.83}$$

According to (5.8), for space inversion, we obtain

$$Z_{\rm D} \Psi_{\rm D}(x) \xrightarrow{P} Z_{\rm D} \Psi^P_{\rm D}(\bar{x}) = Z_{\rm D} \gamma^0 \Psi_{\rm D}(\bar{x}),$$

where $\bar{x} = (x^0, -x^k)$. The matrix $\gamma^5 = \mathrm{diag}\{\sigma^0, -\sigma^0\}$ corresponds to the chirality operator (5.32).

A complex conjugate function corresponds to the charge conjugate state,

$$\overset{*}{f}_{1/2}(x, \mathbf{z}) = -\psi_\alpha(x) \underline{z}^\alpha - \overset{*}{\chi}{}^{\dot\alpha}\,(x)\, \overset{*}{z}_{\dot\alpha},$$

(the minus sign is from the anticommutation of spinors, $\psi_\alpha z^\alpha = -z_\alpha \psi^\alpha$) or in matrix form,

$$Z_{\rm D}\Psi_{\rm D}(x) \xrightarrow{C} \overset{*}{Z}_{\rm D}\overset{*}{\Psi}_{\rm D}(x) = \underline{Z}_{\rm D}\Psi^c_{\rm D}(x),$$
$$\Psi^c_{\rm D}(x) = -\begin{pmatrix}\psi_\alpha(x)\\ \overset{*}{\chi}{}^{\dot\alpha}(x)\end{pmatrix} = i\sigma^2\begin{pmatrix}\psi_\alpha(x)\\ -\overset{*}{\chi}{}^{\dot\alpha}(x)\end{pmatrix}, \tag{5.84}$$

where $\underline{Z}_{\rm D} = (\underline{z}^\alpha, \overset{*}{z}_{\dot\alpha})$, and $Z_{\rm D}$ obey the same transformation law. Thus, we get different scalar functions to describe particles and antiparticles and hence two Dirac equations for the two signs of charge, respectively. This matches completely with the results of Gavrilov and Gitman (2000). It was shown there that in the course of a consistent quantization of a classical model of a spinning particle, such a (charge-symmetric) quantum mechanics appears. At the same time, it is completely equivalent to the one-particle sector of the corresponding quantum field theory.

Real functions $f_{1/2}(x,\mathbf{z}) = \overset{*}{f}_{1/2}(x,\mathbf{z})$ describing Majorana particles depend on the elements of $Z_{\rm M}$ (5.9), and correspondingly $\psi^\alpha(x) = -\chi^\alpha(x) = i\sigma^2\chi_\alpha(x)$. Space reflection maps these functions into functions of $\underline{Z}_{\rm M}$.

For $s = j_1 + j_2 = 1$, we have

$$f_1(x,\mathbf{z}) = \chi_{\alpha\beta}(x)z^\alpha z^\beta + \phi_\alpha^{\dot\beta}(x)z^\alpha\,\overset{*}{\underline{z}}_{\dot\beta} + \psi^{\dot\alpha\dot\beta}(x)\overset{*}{\underline{z}}_{\dot\alpha}\overset{*}{\underline{z}}_{\dot\beta} = \Phi_\mu(x)q^\mu + \frac{1}{2}F_{\mu\nu}(x)q^{\mu\nu}, \tag{5.85}$$

where

$$q^\mu = \frac{1}{2}\sigma^\mu_{\alpha\dot\beta}z^\alpha\overset{*}{\underline{z}}{}^{\dot\beta}, \quad q_\mu q^\mu = 0,$$
$$q_{\mu\nu} = -q_{\nu\mu} = \frac{1}{2}\big[(\sigma_{\mu\nu})_{\alpha\beta}z^\alpha z^\beta + (\bar\sigma_{\mu\nu})_{\dot\alpha\dot\beta}\,\overset{*}{\underline{z}}{}^{\dot\alpha}\overset{*}{\underline{z}}{}^{\dot\beta}\,\big], \tag{5.86}$$

$$\Phi_\mu(x) = -\bar\sigma_\mu^{\dot\beta\alpha}\phi_{\alpha\dot\beta}(x),$$
$$F_{\mu\nu}(x) = -2[(\sigma_{\mu\nu})_{\alpha\beta}\chi^{\alpha\beta}(x) + (\bar\sigma_{\mu\nu})_{\dot\alpha\dot\beta}\psi^{\dot\alpha\dot\beta}(x)]. \tag{5.87}$$

Substituting (5.85) into Eq. (5.72), we obtain

$$m\psi^{\dot\alpha\dot\beta}(x) = \frac{1}{2}\hat p_\mu\bar\sigma^{\mu\dot\alpha\gamma}\phi_\gamma^{\dot\beta}(x), \quad m\chi_{\alpha\beta}(x) = \frac{1}{2}\hat p_\mu\sigma^\mu_{\dot\gamma\alpha}\phi_\beta^{\dot\gamma}(x),$$
$$m\phi_\alpha^{\dot\beta}(x) = \hat p_\mu[\bar\sigma^{\mu\dot\beta\gamma}\chi_{\alpha\gamma}(x) + \sigma^\mu_{\alpha\dot\alpha}\psi^{\dot\alpha\dot\beta}(x)], \tag{5.88}$$

$$mF_{\mu\nu}(x) = \partial_\mu\Phi_\nu(x) - \partial_\nu\Phi_\mu(x), \quad m\Phi_\mu(x) = \partial^\nu F_{\mu\nu}(x). \tag{5.89}$$

The Duffin–Kemmer equation in the form (5.88) or (5.89) is the equation for the irrep $T_{[20]}$ of the $3+2$ de Sitter group and thus for the reducible representation $(1\,0)\oplus(\frac{1}{2}\,\frac{1}{2})\oplus(0\,1)$ of the Lorentz group. This representation contains both the four-vector $\Phi_\mu(x)$ and the antisymmetric tensor $F_{\mu\nu}(x)$, which correspond to chiralities $\lambda = 0$ and $\lambda = \pm 1$. Excluding components $F_{\mu\nu}(x)$, we obtain a second-order system only for the components $\Phi_\mu(x)$ transforming under the irrep $(\frac{1}{2}\,\frac{1}{2})$ of the Lorentz group:

$$(\hat{p}^2 - m^2)\Phi_\mu(x) = 0, \quad \hat{p}^\mu \Phi_\mu(x) = 0. \tag{5.90}$$

One can rewrite the operator $\hat{p}_\mu \hat{\Gamma}^\mu$ in terms of the complex variables $q^\mu$ and $q^{\mu\nu}$,

$$\hat{p}_\mu \hat{\Gamma}^\mu = -i\hat{p}_\mu \big(q^{\mu\nu}\partial/\partial q^\nu - q_\nu \partial/\partial q_{\mu\nu}\big). \tag{5.91}$$

Such a conversion to vector indices is possible when considering any integer spin. Notice that two sets of real spin variables with vector indices can be obtained by substituting elements of $Z_{\mathrm{M}}$ and $Z_{\mathrm{M}}$ instead of $Z_{\mathrm{D}}$ into (5.86).

One may describe the neutral spin-1 field, in particular, by a real function of the elements of the Majorana $z$-spinor, $f_1(x, \mathbf{z}) = \overset{*}{f}_1(x, \mathbf{z})$. However, the spaces of quadratic functions of the Dirac $z$-spinor $Z_{\mathrm{D}}$ and Majorana $z$-spinor $Z_{\mathrm{M}}$ are noninvariant with respect to charge conjugation and space reflection, respectively. To describe a spin-1 neutral particle coinciding with its antiparticle, one may use bilinear functions of $Z_{\mathrm{D}}$ and $\underline{Z}_{\mathrm{D}}$.

For the cases $s = 1/2$ and $s = 1$, the first equation of the system (5.71)–(5.73) (Klein–Gordon equation) is the consequence of the other equations. For $s > 1$, the solutions of (5.72) are characterized by spin and mass spectrum, $s_i = \{s, s-1, \ldots, 1\}$ or $s_i = \{s, s-1, \ldots, 1/2\}$, $m_i = ms/s_i$. Thus, for higher spin fields, the Klein–Gordon equation is an independent condition, allowing us to exclude from the spin spectrum all spins except the maximal $s = j_1 + j_2$.

The cases $s = 1/2$ and $s = 1$ are also exceptions in the sense of simplicity of labeling the components by spinor or vector indices. The number of indices of symmetric spin-tensors necessary for labeling higher spin components increases in spite of the fact that it is sufficient to use only three operators and therefore only three numbers for labeling the states belonging to symmetric irreps of $SO(3, 2)$.

In particular, for a spin-3/2 particle, there exist four kinds of components, namely $\psi_{\alpha\beta\gamma}$, $\psi_{\dot\alpha\beta\gamma}$, $\psi_{\dot\alpha\dot\beta\gamma}$, $\psi_{\dot\alpha\dot\beta\dot\gamma}$, corresponding to four possible values of the chirality. For a spin-2 particle, the representation in terms of $q^\mu$ and $q^{\mu\nu}$ is cumbersome,

$$f_2(x, q) = \Phi_{\mu\nu}(x) q^\mu q^\nu + \frac{1}{2} F_{\mu\nu,\rho}(x) q^{\mu\nu} q^\rho + \frac{1}{4} F_{\mu\nu,\rho\sigma}(x) q^{\mu\nu} q^{\rho\sigma}, \tag{5.92}$$

with the necessity to fix independent components by means of relations $q_\mu q^\mu = 0$, $q_{\mu\nu} q^\mu + q_{\mu\nu} q^\nu = 0$, and so on.

Thus, beginning from spin 3/2, it is convenient to use the universal notation $\psi^{m_1 m_2}_{j_1 j_2}(x)$ associated with the decomposition (5.20) over the monomial chiral basis (5.22) [see also (B11)–(B13)]. Two indices $j_1$, $j_2$ label spin $s = j_1 + j_2$ and chirality $\lambda = j_1 - j_2$ and two indices $m_1, m_2$ label independent components inside the irrep of the Lorentz group. This notation is also suitable for infinite-dimensional representations.

By analogy with the $2+1$ case, one can find plane wave solutions of the system (5.71)–(5.72) for any spin s in general form without using the matrix representation. Corresponding to the states of a particle moving along $x^3$ are eigenstates of the operator $\hat{p}_i\hat{S}^{\mathrm{i}}$ with eigenvalues $|p|\sigma$, where $\sigma = s_3\,\mathrm{sign}\,p_3$ is the helicity. These states have the form

$$f_{m,s,\sigma}(x,\mathbf{z}) = \sum_{\sigma=-s}^{s} C_\sigma e^{ik_0x^0+k_3x^3}(z^1e^a + \overset{*}{\underline{z}}_{\dot{1}}e^{-a})^{s+\sigma}(z^2e^{-a} + \overset{*}{\underline{z}}_{\dot{2}}e^{a})^{s-\sigma}$$
$$+ \sum_{\sigma'=-s}^{s} C_{\sigma'}e^{-ik_0x^0-k_3x^3}(z^1e^a - \overset{*}{\underline{z}}_{\dot{1}}\,e^{-a})^{s-\sigma}(z^2e^a - \overset{*}{\underline{z}}_{\dot{2}}\,e^{-a})^{s+\sigma}, \quad (5.93)$$

where $e^a$ is given by (5.76). For a rest particle, one can obtain the general solution characterized by the spin projection $s'$ in the directions of $\mathbf{n}$ from (5.74) by the rotation $z'_\beta = U^\alpha_\beta z_\alpha$, $U \subset SU(2)$. For a particle characterized by a momentum direction $\mathbf{n}$ and helicity $\sigma$, starting from the state (5.93), one can obtain the solution by an analogous rotation.

The improper Poincaré group includes space reflection, which interchanges the representations $(j_1, j_2)$ and $(j_2, j_1)$. Therefore, we consider the equations connecting these representations [and the corresponding components of the solutions of the system (5.71)–(5.73)] in more detail.

In the case $j_1 = j_2$, solutions of the system (5.71)–(5.73) are characterized by fixed spin $s = j_1 + j_2$ and mass $m$. Thus, the relations (5.27)–(5.30) are valid, and corresponding $2(2s+1)$ components obey the equations for a massive tensor field (5.44).

In the general case, the equations connecting the components transforming under irreps $(j_1, j_2)$ and $(j_2, j_1)$ of the Lorentz subgroup have the form

$$(2j_2)!\,\left(\hat{p}_\mu\hat{V}^\mu_{12}\right)^{2|\lambda|} f_{j_1j_2}(x,z,\overset{*}{\underline{z}}) = (2j_1)!\,m^{2|\lambda|}\,f_{j_2j_1}(x,z,\overset{*}{\underline{z}}),$$
$$(2j_1)!\,\left(\hat{p}_\mu\hat{V}^\mu_{21}\right)^{2|\lambda|} f_{j_2j_1}(x,z,\overset{*}{\underline{z}}) = (2j_2)!\,m^{2|\lambda|}\,f_{j_1j_2}(x,z,\overset{*}{\underline{z}}), \quad (5.94)$$

where $j_1 > j_2$, $|\lambda| = j_1 - j_2$. These equations are invariant under space reflection. Using the decomposition (5.79) and the explicit form of the general solution (5.74) of the system (5.71)–(5.73) in the rest frame, one can prove the validity of (5.94) by direct calculation. Going over to spin-tensor notation, we get

$$p_\mu\bar{\sigma}^{\mu\dot{\alpha}_{2j_2+1}\alpha_{2j_2+1}}\cdots p_\nu\bar{\sigma}^{\nu\dot{\alpha}_{2j_1}\alpha_{2j_1}}\,\psi^{\dot{\alpha}_1\cdots\dot{\alpha}_{2j_2}}_{\alpha_1\cdots\alpha_{2j_1}}(x) = m^{2|\lambda|}\psi^{\dot{\alpha}_1\cdots\dot{\alpha}_{2j_1}}_{\alpha_1\cdots\alpha_{2j_2}}(x),$$
$$p_\mu\sigma^\mu_{\alpha_{2j_2+1}\dot{\alpha}_{2j_2+1}}\cdots p_\nu\sigma^\nu_{\alpha_{2j_1}\dot{\alpha}_{2j_1}}\,\psi^{\dot{\alpha}_1\cdots\dot{\alpha}_{2j_1}}_{\alpha_1\cdots\alpha_{2j_2}}(x) = m^{2|\lambda|}\psi^{\dot{\alpha}_1\cdots\dot{\alpha}_{2j_2}}_{\alpha_1\cdots\alpha_{2j_1}}(x). \quad (5.95)$$

Equations (5.95) are consequences of the system (5.71)–(5.73), but unlike this system, in the general case, they require supplementary conditions to fix mass and spin.

Equations (5.95) are first-order equations only in the case $|\lambda| = 1/2$, which corresponds to representations $(\frac{2j\pm1}{4}\ \frac{2j\mp1}{4})$, $j = j_1 + j_2$, describing half-integer spins. In this case, going over to vector indices and supplementing the equations by subsidiary conditions (5.45) [which also are consequences of the system (5.71)–(5.73) and exclude components with $s < j_1 + j_2$], we obtain the Rarita–Schwinger equations (Rarita and Schwinger, 1941)

$$(\hat{p}_\mu \gamma^\mu - m)\Psi_{\mu_1\mu_2\cdots\mu_n}(x) = 0, \quad \gamma^\mu \Psi_{\mu\mu_2\cdots\mu_n}(x) = 0, \tag{5.96}$$

where $n = 2s - 1$ and $\Psi_{\mu_1\cdots\mu_n}(x)$ is a four-component column composed of $\Psi_{\mu_1\cdots\mu_n\alpha}(x)$ and $\Psi^{\dot\alpha}_{\mu_1\cdots\mu_n}(x)$. The conditions $\partial^\mu \Psi_{\mu\mu_2\cdots\mu_n}(x) = 0$ and $\Psi^\mu_{\mu\cdots\mu_n}(x) = 0$ appear as consequences of these two equations (Ohnuki, 1988).

The case $|\lambda| = s$ corresponding to representations $(s\ 0)$ and $(0\ s)$ is preferred because of the minimal number of components. In this case, Eq. (5.95) are $2s$-order Joos–Weinberg equations (Joos, 1962; Weinberg, 1964, 1969) of so-called $2(2s + 1)$-component theory,

$$\begin{aligned} p_\mu \bar{\sigma}^{\mu\dot\alpha_1\alpha_1} \cdots p_\nu \bar{\sigma}^{\nu\dot\alpha_{2s}\alpha_{2s}} \psi_{\alpha_1\cdots\alpha_{2s}}(x) &= m^{2s} \psi^{\dot\alpha_1\cdots\dot\alpha_{2s}}(x), \\ p_\mu \sigma^\mu_{\alpha_1\dot\alpha_1} \cdots p_\nu \sigma^\nu_{\alpha_{2s}\dot\alpha_{2s}} \psi^{\dot\alpha_1\cdots\dot\alpha_{2s}}(x) &= m^{2s} \psi_{\alpha_1\cdots\alpha_{2s}}(x). \end{aligned} \tag{5.97}$$

In the rest frame, as a consequence, we obtain $p_0^{4s} = m^{4s}$, and for $s \geq 1$, the Joos–Weinberg equations have solutions with complex energy $p_0$, $|p_0| = m$. The existence of such solutions was pointed out also in Ahluwalia and Ernst (1992).

## 5.4. Relativistic Wave Equations Invariant Under the Improper Poincaré Group. Equations for Several Scalar Functions

We have considered the linear equations for *one* scalar function on the group. The condition of invariance under space reflection leads to the system (5.71)–(5.73) for a particle with spin $s = j_1 + j_2$ and mass $m$.

For the construction of invariant wave equations, one may also use the operators $\hat{p}_\mu \hat{V}^\mu_{ik}$, which are not invariant under space reflections. Using several scalar functions $f(x, \mathbf{z})$, it is possible to restore the invariance under space reflections.

In particular, Eqs. (5.55) containing operators $\hat{V}^\mu_{12(k)}$ and $\hat{V}^\mu_{21(k)}$ connect two scalar functions. Using the decomposition (5.23) in terms of spin-tensors, we obtain the Dirac–Fierz–Pauli equations (Dirac, 1936; Fierz and Pauli, 1939),

$$\begin{aligned} \hat{p}_\mu \bar{\sigma}^{\mu\dot\alpha\beta} \psi^{\dot\alpha_1\cdots\dot\alpha_l}_{\beta\beta_1\cdots\beta_n}(x) &= \varkappa \psi^{\dot\alpha\dot\alpha_1\cdots\dot\alpha_l}_{\beta_1\cdots\beta_n}(x) \\ \hat{p}_\mu \sigma^\mu_{\beta\dot\alpha} \psi^{\dot\alpha\dot\alpha_1\cdots\dot\alpha_l}_{\beta_1\cdots\beta_n}(x) &= \varkappa \psi^{\dot\alpha_1\cdots\dot\alpha_l}_{\beta\beta_1\cdots\beta_n}(x). \end{aligned} \tag{5.98}$$

These equations connect two functions transforming under irreps $(\frac{n}{2} + \frac{1}{2}\ \frac{l}{2})$ and $(\frac{n}{2}\ \frac{l}{2} + \frac{1}{2})$ of the Lorentz group and for $n = l$ map to one another under parity transformation.

Let us consider a system of equations of the form (5.55), (5.56), which connect several scalar functions with different $j_1, j_2$. The equations of this system connect the representation $(j_1, j_2)$ with at least one of the representations $(j_1 \pm 1, j_2 \mp 1)$, $(j_1 \pm 1, j_2 \pm 1)$. This allows one to identify this system with the general Gel'fand–Yaglom equations (Gel'fand and Yaglom, 1948; Gel'fand *et al.*, 1963)

$$(\alpha^\mu \hat{p}_\mu - \varkappa)\psi = 0, \quad [S^{\lambda\mu}, \alpha^\nu] = i(\eta^{\mu\nu}\alpha^\lambda - \eta^{\lambda\nu}\alpha^\mu). \tag{5.99}$$

In the present approach, the latter relation is a consequence of the commutation relations $[\hat{S}^{\lambda\mu}, \hat{V}^\nu_{ik}] = i(\eta^{\mu\nu}\hat{V}^\lambda_{ik} - \eta^{\lambda\nu}\hat{V}^\mu_{ik})$. This relation is necessary for Poincaré invariance of the equations (Barut and Raczka, 1977; Gel'fand *et al.*, 1963).

Supplemented by the commutation relations $[\alpha^\mu, \alpha^\nu] = S^{\mu\nu}$, finite-component equations of the form (5.99) are known as Bhabha equations (Bhabha, 1945), although for they were first systematically considered by Lubanski (1942). These equations are classified according to the finite-dimensional irreps of the $3 + 2$ de Sitter group $SO(3, 2)$. Other possible commutation relations of the matrices $\alpha^\mu$ are discussed in Castell (1967).

Equation (5.72) considered on a scalar function is a particular case of the Bhabha equations. This case corresponds to symmetric irreps $T_{[2s0]}$ of the $3 + 2$ de Sitter group. Generally speaking, the Bhabha equations are characterized by a finite number of different $m$ and $s$. Therefore, these equations connect fields transforming under nonequivalent irreps of the Poincaré group.

If the equations include the operators $\hat{p}_\mu \hat{V}^\mu_{11}$ and $\hat{p}_\mu \hat{V}^\mu_{22}$, then either the equations describe at least two different spins $s$ or the condition $s = j_1 + j_2$ connecting spin $s$ with a highest weight of the irrep of the Lorentz group is not valid.

We cite as an example the system connecting irreps (0 0) and $(\frac{1}{2}\ \frac{1}{2})$ of the Lorentz group:

$$\hat{p}_\mu \hat{V}^\mu_{11} f_{00}(x, \mathbf{z}) = \varkappa_1 f_{\frac{1}{2}\frac{1}{2}}(x, \mathbf{z}), \quad \hat{p}_\mu \hat{V}^\mu_{22} f_{\frac{1}{2}\frac{1}{2}}(x, \mathbf{z}) = \varkappa_2 f_{00}(x, \mathbf{z}) \tag{5.100}$$

where $f_{00}(x, \mathbf{z}) = \psi(x)$, $f_{\frac{1}{2}\frac{1}{2}}(x, \mathbf{z}) = \psi^{\dot\beta}_\alpha(x) z^\alpha \overset{*}{\underline{z}}_{\dot\beta}$; in componentwise form, we have $\hat{p}_\mu \psi = 2\varkappa_1 \psi_\mu$, $\hat{p}_\mu \psi^\mu = \varkappa_2 \psi$. In the rest frame, we obtain $\varkappa_2 = 2\varkappa_1 = m$. Thus, the system (5.100) is equivalent to the Duffin equation for scalar particles, which corresponds to the five-dimensional vector irrep $T_{[01]}$ of the $SO(3, 2)$ group.

## 5.5. Relativistic Wave Equations Invariant Under the Improper Poincaré Group. Equations for Particles with Composite Spin

Many-particle systems are described by functions of the sets of variables $x_{(i)}, z_{(i)}, \overset{*}{\underline{z}}_{(i)}$. Here we will consider not many-particle systems in the usual sense, but objects corresponding to functions $f(x, z_{(1)}, \overset{*}{\underline{z}}_{(1)}, \ldots, z_{(n)}, \overset{*}{\underline{z}}_{(n)})$ [or, briefly, $f(z, \{\mathbf{z}_{(i)}\})$], that is, to functions of one set of $x$ and several sets of $\mathbf{z}$. One may interpret these objects as particles with composite spin.

As an example, we will consider the Ivanenko–Landau–Kähler (or Dirac–Kähler) equation (Ivanenko and Landau, 1928; Kähler, 1962). Let us write scalar function $f(x, \mathbf{z}_{(1)}, \mathbf{z}_{(2)})$ linear in $\mathbf{z}_{(1)}$ and $\mathbf{z}_{(2)}$ in the form

$$f\big(x, \mathbf{z}_{(1)}, \mathbf{z}_{(2)}\big) = Z_{\mathrm{D}}^{(1)} \Psi(x) \big(Z_{\mathrm{D}}^{(2)}\big)^{\dagger} = \sum_{i,j=1}^{4} \big(Z_{\mathrm{D}}^{(1)}\big)_i \Psi_{ij}(x) \big(\overset{*}{Z}{}_{\mathrm{D}}^{(2)}\big)_j, \quad (5.101)$$

where $Z_{\mathrm{D}} = (z^1 z^2\ \overset{*}{\underline{z}}_{\dot 1} \overset{*}{\underline{z}}_{\dot 2})$, and $\Psi(x)$ is a $4 \times 4$ matrix with a transformation rule

$$\Psi'(x') = \breve{U} \Psi(x) (\breve{U})^{\dagger}, \quad \breve{U} = \operatorname{diag}\{U, (U^{-1})^{\dagger}\},$$

in contrast to the transformation rule $\Psi'_{\mathrm{D}}(x') = \breve{U} \Psi_{\mathrm{D}}(x)$ of the Dirac spinor (5.83). Let us impose the equation on the first ("left") spin subsystem,

$$\big(\hat{p}_\mu \hat{\Gamma}^{\mu}_{(1)} - m/2\big) f(x, \mathbf{z}_{(1)}, \mathbf{z}_{(2)}) = 0, \quad (5.102)$$

and we do not impose any conditions on the second ("right") spin subsystem. Writing (5.102) in componentwise form, we obtain the Ivanenko–Landau–Kähler equation in spinor matrix representation

$$(\hat{p}_\mu \gamma^\mu - m) \Psi(x) = 0. \quad (5.103)$$

According to (5.103), the 16 components $\Psi_{ij}(x)$ obey the Klein–Gordon equation, and therefore the mass is equal to $m$. The spin of both subsystems is equal to $1/2$. The spin of the system is indefinite, and there are both spin-0 and spin-1 components.

The consideration of this equation is associated mainly with attempts to describe fermions by antisymmetric tensor fields [see, e.g., Benn and Tucker (1983, 1988), Bullinaria (1986), Obukhov and Solodukhin (1993), and also Ivanenko *et al.* (1985) as a good introduction]. The spin subsystems ("left-spin" and "right-spin") were considered in Benn and Tucker (1983) and Obukhov and Solodukhin (1993).

Let us consider now linear symmetric functions of $\mathbf{z}_{(1)}, \ldots, \mathbf{z}_{(n+l)}$:

$$f_{\frac{n}{2}\frac{l}{2}}\big(x, \{\mathbf{z}_{(i)}\}\big) = \psi^{\dot\alpha_1,\ldots,\dot\alpha_l}_{\beta_1,\ldots,\beta_n}(x) \sum z^{\beta_1}_{(1)} \cdots z^{\beta_n}_{(n)}\ \overset{*}{\underline{z}}_{(n+1)\dot\alpha_1} \cdots \overset{*}{\underline{z}}_{(n+l)\dot\alpha_l}, \quad (5.104)$$

where the symmetric spinors $\psi^{\dot\alpha_1,\ldots,\dot\alpha_l}_{\beta_1,\ldots,\beta_n}(x)$ transform under the irreps $(n/2, l/2)$, and all permutations of $1, \ldots, n+l$ are summed over. As a consequence of the symmetry of the multispinors with respect to index permutations, spin subsystems are indistinguishable, and this allows us to use functions of several sets of spin variables for describing the usual particles.

One obtains the Dirac–Fierz–Pauli equations (5.98) by acting by the operators $\hat{V}^{\mu}_{12(k)}$ and $\hat{V}^{\mu}_{21(k)}$ on the functions (5.104) corresponding to irreps $(\frac{n}{2}+\frac{1}{2}, \frac{l}{2})$ and $(\frac{n}{2}, \frac{l}{2}+\frac{1}{2})$ of the Lorentz group:

$$\hat{V}^{\mu}_{12(k)} f_{\frac{n}{2}+\frac{1}{2},\frac{l}{2}}\big(x, \{\mathbf{z}_{(i)}\}\big) = \varkappa f_{\frac{n}{2},\frac{l}{2}+\frac{1}{2}}\big(x, \{\mathbf{z}_{(i)}\}\big),$$
$$\hat{V}^{\mu}_{21(k)} f_{\frac{n}{2},\frac{l}{2}+\frac{1}{2}}\big(x, \{\mathbf{z}_{(i)}\}\big) = \varkappa f_{\frac{n}{2}+\frac{1}{2},\frac{l}{2}}\big(x, \{\mathbf{z}_{(i)}\}\big). \quad (5.105)$$

In the general case, a linear symmetric function of $\mathbf{z}_{(k)}$, $k = 1, \ldots, 2j$, has the form

$$f_j(x, \{\mathbf{z}_{(i)}\}) = \sum_{n,l;n+l=2j} \psi^{\dot{\alpha}_1,\ldots,\dot{\alpha}_l}_{\beta_1\ldots\beta_n}(x)\, z^{\beta_1}_{(1)} \ldots z^{\beta_n}_{(n)} \overset{*}{\underline{z}}_{(n+1)\dot{\alpha}_1} \ldots \overset{*}{\underline{z}}_{(n+l)\dot{\alpha}_l}. \quad (5.106)$$

and corresponds to the symmetric part of the representation $((\frac{1}{2}\ 0) \oplus (0, \frac{1}{2}))^{2j}$. This symmetric part expands into a direct sum of irreps $(j_1, j_2)$, $j_1 + j_2 = j$. We impose on each spin subsystem the condition

$$\left(\hat{p}_\mu \hat{\Gamma}^\mu_{(k)} - m/2\right) f\left(x, \mathbf{z}_{(1)}, \ldots, \mathbf{z}_{(2j)}\right) = 0, \quad k = 1, \ldots, 2j. \quad (5.107)$$

Rewriting this equations in four-component form, we obtain the Bargmann–Wigner equations (Bargmann and Wigner, 1948; Greiner, 1997; Ohnuki, 1988)

$$\left(\hat{p}_\mu \gamma^\mu_{(k)} - m\right)_{\alpha_k\beta_k} \psi_{\beta_1\ldots\beta_k\ldots\beta_{2j}}(x) = 0. \quad (5.108)$$

As a consequence of (5.107), we obtain the equations for the system as a whole:

$$(\hat{p}^2 - m^2) f\left(x, \mathbf{z}_{(1)}, \ldots, \mathbf{z}_{(2j)}\right) = 0,$$

$$\left(\hat{p}_\mu \hat{\Gamma}^\mu - ms\right) f\left(x, \mathbf{z}_{(1)}, \ldots, \mathbf{z}_{(2j)}\right) = 0, \quad \hat{\Gamma}^\mu = \sum \hat{\Gamma}^\mu_{(k)}, \quad (5.109)$$

which are analogous to Eqs. (5.71)–(5.72) for the case $s = j_1 + j_2$. Both the Bargmann–Wigner equations and system (5.109) have $2(2s + 1)$ independent solutions $\psi(x)$, and therefore these systems are equivalent.

## 5.6. Relativistic Wave Equations: Comparative Consideration

In the framework of the group-theoretic classification of the scalar fields $f(x, \mathbf{z})$ on the Poincaré group, we have obtained two types of equations describing unique spin and mass, namely equations for the eigenfunctions of the Casimir operator of the Lorentz spin subgroup [$j_1$ and $j_2$ are fixed; see (5.30)] and equations for the eigenfunctions of the Casimir operator of the $SO(3, 2)$ group (the sum $j_1 + j_2$ is fixed). Below we will consider comparative characteristics of these equations and also the case $(j_1\ j_2) \oplus (j_2\ j_1)$ corresponding to irreps of the improper Poincaré group, but requiring two scalar functions for its formulation.

1. Equations for the functions corresponding to the fixed irrep $(j_1\ j_2)$ of the Lorentz group. Mass and spin irreducibility conditions leave $2(2s + 1)$ independent components corresponding to two improper Poincaré group irreps differing in the sign of $p_0$. For $s = j_1 + j_2$, the equations in spin-tensor form constitute the system of the Klein–Gordon equation and the subsidiary condition (5.42), which eliminates components with other possible values of spin $s$ for fixed $j_1, j_2 | j_1 - j_2| \leq s < j_1 + j_2$. For $s \neq j_1 + j_2$, one should consider the general subsidiary condition (5.43). An alternative to the use of the subsidiary condition is to consider functions of momentum and spin variables with invariant constraints (5.39).

There are two preferred cases. The first corresponds to the representations $(\frac{s}{2}\ \frac{s}{2})$ mapping onto themselves under space reflection and are most often used to describe integer spins. The second corresponds to the representations $(s\ 0)$ and $(0\ s)$. In this case, there is no necessity to impose subsidiary conditions since they are fulfilled identically.

2. Equations for functions corresponding to the representations $(j_1\ j_2)$ and $(j_2\ j_1)$, $j_1 \neq j_2$, which are interchanged under space reflection. Unlike the equations considered above for fixed $j_1$, $j_2$, these equations in the general case are not formulated as equations for one scalar function $f(x, \mathbf{z})$. The conditions of mass and spin irreducibility leave $4(2s + 1)$ independent components corresponding to four improper Poincaré group irreps differing in the sign of $p_0$ and intrinsic parity $\eta$. To choose $2(2s + 1)$ components corresponding to fixed sign of $\eta$ or $p_0\eta$, it is necessary to supplement these conditions by Eqs. (5.94) connecting components corresponding to $(j_1\ j_2)$ and $(j_2\ j_1)$.

Equations (5.94) are first-order equations only for the representations $(j + \frac{1}{2}\ j) \oplus (j\ j + \frac{1}{2})$. These representations and the associated Rarita–Schwinger equations (5.96) are most often used to describe half-integer spins. However, just as in the case of representations $(j\ j)$, subsidiary conditions supplement the field equations, and the number of equations exceeds the number of field components. Therefore, one has an overdetermined set of equations, which, although consistent in the free-field case, for $s > 1$ becomes self-contradictory with minimal electromagnetic coupling (Fierz and Pauli, 1939). In order to avoid inconsistency, it is possible to give a Lagrangian formulation, introducing auxiliary fields (Fierz and Pauli, 1939; Singh and Hagen, 1974a,b), but this formulation leads to acasual propagation with minimal electromagnetic coupling (Capri and Kobes, 1980; Tung, 1967; Velo, 1972; Velo and Zwanziger, 1969; Wightman, 1978; Zwanziger, 1978).

For the case $(s\ 0) \oplus (0\ s)$, one can construct the $2(2s + 1)$-component theory, but the corresponding Joos–Weinberg equations of order $2s$ (Joos, 1962; Weinberg, 1964) [see (5.97)] for $s \geq 1$ also have solutions with complex energy.

The second-order equation for the representation $(s\ 0) \oplus (0\ s)$, $[\hat{p}^2 - \frac{e}{2s}\hat{S}^{\mu\nu} \times F_{\mu\nu} - m^2]\psi(x) = 0$ (Feynman and Gell-Mann, 1958; Hurley, 1971, 1974; Ionesco-Pallas, 1967), for a free particle possesses $4(2s + 1)$ independent components differing in spin projection and in signs of $p_0$ and $\eta$. On the other hand, this equation describes unique mass and spin and is characterized by casual solutions. In particular, exact solutions in an external constant uniform electromagnetic field are known (Kruglov, 1999). One may rewrite the above equation as a first-order equation with minimal coupling for representations $(s\ 0) \oplus (s - \frac{1}{2}\ \frac{1}{2}) \oplus (\frac{1}{2}\ s - \frac{1}{2}) \oplus (0\ s)$. As noted in Tung (1967), this is the simplest class of representations describing unique mass and spin, which led to first-order equations without subsidiary conditions.

3. Equations (5.71)–(5.72) for eigenfunctions of the Casimir operator (5.73) of the $SO(3, 2)$ group with eigenvalues $4s(s+2)$, $s = j_1 + j_2$:

$$(\hat{p}_\mu \hat{\Gamma}^\mu - ms) f(x, z, \overset{*}{\underline{z}}) = 0, \quad (\hat{p}^2 - m^2) f(x, z, \overset{*}{\underline{z}}) = 0. \tag{5.110}$$

The condition of spin irreducibility (5.51) is a consequence of this system.

The first equation of the system is the Bhabha equation (Bhabha, 1945; Lubanski, 1942) corresponding to the symmetric irrep $T_{[2s\ 0]}$ of the group $Sp(4, R) \sim SO(3, 2)$. This equation represent a straightforward higher spin generalization of the Dirac and spin-1 Duffin–Kemmer equations. Both the Bhabha equations and the problem of minimal coupling for these equations were considered by Krajcik and Nieto [see Krajcik and Nieto (1977), which contains references to six earlier papers]. The theory is casual with minimal electromagnetic coupling (Krajcik and Nieto, 1976), but in the general case, the Bhabha equations describe multimass systems. The connection of the Rarita–Schwinger and Bargmann–Wigner equations with the Bhabha equations also was considered in Loide *et al.* (1997).

The solutions of the system (5.110) have components transforming under $2s + 1$ irreps $(j_1, j_2)$, $j_1 + j_2 = s$, of the Lorentz group. But the components corresponding to different chiralities $\lambda = j_1 - j_2$ are not independent. In contrast to the left generators of the Poincaré group, the operators $\hat{\Gamma}_\mu$ do not commute with the chirality operator (which is the right generator of the Poincaré group) and combine $2s + 1$ representations of the Lorentz group into one irrep of the $3 + 2$ de Sitter group $SO(3, 2)$.

The current component $j^0$ is positive definite for half-integer-spin particles and the energy density is positive definite for integer-spin particles (see Appendix B).

In the rest frame, Eqs. (5.110) have $2s + 1$ positive- and $2s + 1$ negative-frequency solutions labeled by different spin projections [see (5.74)] and half-integer-spin solutions with opposite frequency are characterized by opposite parity. In the ultrarelativistic limit, two solutions with opposite sign of $p_0$ correspond to any of $2s + 1$ of possible values of chirality [see (5.78)].

Thus, the system (5.110) describes a particle with unique spin and mass, is invariant under parity transformation, and possesses $2(2s + 1)$ independent components.

Let us briefly consider the problem of equivalence of the different RWE. In the case of free fields, using the relation

$$[\partial_\mu, \partial_\nu] = 0, \tag{5.111}$$

one can establish the equivalence of wide class of RWE.

As established above, in the free case, the system (5.110) and the Bargmann–Wigner equations (5.107), which both describe a particle by means of wave functions with componets transforming under $2s+1$ irreps $(j_1 j_2)$, $j_1 + j_2 = s$, of the Lorentz group, are equivalent. However, the formulation (5.110) is more general since, unlike the Bargmann–Wigner equations, it can be considered also in the case of infinite-dimensional unitary representations of the Lorentz group, as is done above with an analogous system in the $(2+1)$-dimensional case.

The above free equations for representations $(j_1\ j_2)$ or $(j_1\ j_2) \oplus (j_2\ j_1)$ can be obtained as a consequence of the Bargmann-Wigner equations (Greiner, 1997; Ohnuki, 1988) or the system (5.110) by excluding other components. In the general case for $m \neq 0$, one may express all components in terms of the components corresponding to two chiralities $\pm\lambda$, where $-s \leq \lambda \leq s$.

It is obvious that a coupling which is minimal for one system is not minimal for another, "equivalent" system if one uses the relation (5.111) to prove this equivalence in the free case. These equations will differ by terms proportional to the commutator of covariant derivatives $[D_\mu, D_\nu] = igF_{\mu\nu}$.

Therefore, when an interaction is introduced, the system of equations can be found to be inconsistent if, when taking account of (5.111), some equations are consequences of others. In particular, the spin-1 Bargmann-Wigner equations with minimal electromagnetic coupling are inconsistent [for the proof, see, e.g., Buchbinder and Shavartsman (1993)], but the Duffin–Kemmer and Proca equations with minimal coupling, which are equivalent to them in the free case, are consistent and characterized by causal solutions (Velo and Zwanziger, 1969).

Recently, different approaches have been considered to introduce interactions for higher spin massive fields [see, in particular, Buchbinder *et al.* (1999, 2000a), Klishevich (2000), and Kruglov (1999)]. We hope the present approach will offer new possibilities to describe interacting higher spin fields.

## 6. EQUATIONS FOR FIXED SPIN AND MASS: GENERAL FEATURES

Consider now the general properties of the obtained equations describing a particle with unique mass $m > 0$ and spin $s$ in two dimensions,

$$\hat{p}^2 f(x,\theta) = m^2 f(x,\theta), \tag{6.1}$$

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x,\theta) = msf(x,\theta), \tag{6.2}$$

in three dimensions,

$$\hat{p}^2 f(x,\mathbf{z}) = m^2 f(x,\mathbf{z}), \tag{6.3}$$

$$\hat{p}_\mu \hat{S}^\mu f(x,\mathbf{z}) = msf(x,\mathbf{z}), \tag{6.4}$$

$$\hat{S}_\mu \hat{S}^\mu f(x,\ \mathbf{z}) = S(S+1) f(x,\mathbf{z}), \tag{6.5}$$

and in four dimensions,

$$\hat{p}^2 f(x, \mathbf{z}) = m^2 f(x, \mathbf{z}), \tag{6.6}$$

$$\hat{p}_\mu \hat{\Gamma}^\mu f(x, \mathbf{z}) = msf(x, \mathbf{z}), \tag{6.7}$$

$$\hat{S}_{ab}\hat{S}^{ab} f(x, \mathbf{z}) = 4S(S+2) f(x, \mathbf{z}). \tag{6.8}$$

In the latter case, we suppose in addition $s = \pm S$ to avoid nontrivial spin and mass spectrum.

In all dimensions, the first equation (condition of mass irreducibility) is the eigenvalue equation for the Casimir operator of the Poincaré group. The other equations, although they seem similar, have different origins in even and odd dimensions. This is related to the different role of space inversion.

In $2+1$ dimensions, the other equations (6.4)–(6.5) are eigenvalue equations for the Casimir operator of the Poincaré group and the spin Lorentz subgroup.

In even dimensions, the Casimir operators of the Lorentz subgroup do not commute with the space inversion operator, and space inversion combines two equivalent representations of the proper Poincaré group labeled by chiralities $\pm\lambda$ into representations of the improper Poincaré group. If one rejects equations that fix chirality [in $3+1$ dimensions, this corresponds to the transition to the system (5.50)–(5.52)], then in the rest frame, it is easy to see that there is a redundant number of independent components. Thus, it is necessary to construct an equation connecting the states with different chiralities, and a corresponding new set of commuting operators. This can be done by using supplementary operators $\hat{\Gamma}^\mu$, which extend the Lorentz group $SO(D,1)$ up to the $SO(D,2)$ group with the maximal compact subgroup $SO(D) \otimes SO(2)$. The operator $\hat{\Gamma}^0$ is the generator of the compact $SO(2)$ subgroup.

The third equation of the system fixes the power $2S$ of homogeneity of the functions $f(x, \mathbf{z})$ in $\mathbf{z}$ and therefore fixes the irrep of the Lorentz group in $2+1$ dimensions or of the $3+2$ de Sitter group in $3+1$ dimensions. [In $1+1$ dimensions, there exists an analogous equation $\hat{\Gamma}_a \hat{\Gamma}^a f(x, \theta) = s(s+1) f(x, \theta)$, but, in fact, this equation defines the structure of $\hat{\Gamma}^\mu$.]

The positive (half-) integer $S = s$ correspond to finite-dimensional nonunitary irreps of the Lorentz (or de Sitter) group. Such irreps are realized in the space of power $2s$ polynomials in $\mathbf{z}$.

Negative $S = -s$ correspond to infinite-dimensional unitary irreps. The unitary property allows one to combine the probability amplitude interpretation and relativistic invariance [the desirability of this combination was stressed by Dirac (1972b)]. Thus, the equations under consideration allow two approaches to the description of the same spin by means of both finite-dimensional nonunitary and infinite-dimensional unitary irreps.

In $1+1$ and $2+1$ dimensions, there is the possibliity of the existence of particles with fractional spin since the groups $SO(1,1)$ and $SO(2,1)$ do not contain a compact Abelian subgroup. However, the description of massive particles with fractional spin can be given only in terms of the infinite-dimensional irreps of the group $SO(2,1)$. This is another reason to consider infinite-dimensional irreps.

Fixing the irrep of the Lorentz (or de Sitter) group with the help of the third equation of the system, one can come to the usual multicomponent matrix description by the separation of space and spin variables: $f(x,\mathbf{z}) = \sum \phi_n(\mathbf{z})\psi_n(x)$, where $\phi_n(\mathbf{z})$, form the basis in the representation space of the Lorentz (or de Sitter) group. Thus, depending on the choice of the solution of the third equation, the second equation in the matrix representation is either a finite-component equation or an infinite-component equation of Majorana type.

For fundamental spinor irreps, the action of differential operators $2\hat{S}^\mu$ in $2+1$ dimensions and $2\hat{\Gamma}^\mu$ in $1+1$ and $3+1$ dimensions in the space of functions $f(x,\mathbf{z})$ on the Poincaré group can be rewritten in terms of the action of corresponding $\gamma$-matrices on the functions $\psi(x)$.

Differential operators $\hat{\Gamma}^\mu$ and matrices $\gamma^\mu/2$ obey the same commutation relations

$$[\hat{\Gamma}^\mu, \hat{\Gamma}^\nu] = -i\hat{S}^{\mu\nu}, \quad [\hat{S}^\mu, \hat{S}^\nu] = -i\epsilon^{\mu\nu\rho}\hat{S}_\rho.$$

In $3+1$ dimensions, the operators $\hat{\Gamma}^\mu$ and $\hat{S}^{\mu\nu}$ obey the commutation relations of generators of the $SO(3,2)$ group [see (5.62)].

Anticommutation relations for the operators $\hat{S}^\mu$ in $2+1$ and $\hat{\Gamma}^\mu$ in $1+1$ and $3+1$ dimensions are analogous to the relations for $\gamma$-matrices,

$$[\hat{S}^\mu, \hat{S}^\nu]_+ = \frac{1}{2}\eta^{\mu\nu}, \quad [\hat{\Gamma}^\mu, \hat{\Gamma}^\nu]_+ = \frac{1}{2}\eta^{\mu\nu},$$

and are valid only for fundamental spinor irreps. This is a group-theoretic property connected with the fact that for these irreps, the double action of lowering or raising operators on any state gives zero as a result. [Notice that, besides the case of spinor irreps of orthogonal groups, anticommutation relations also hold for fundamental $N$-dimensional irreps of $Sp(N)$ and $SU(N)$ groups (Gitman and Shelepin, 1998).]

For $s = 1/2$ and $s = 1$, the first equation of the system (condition of mass irreducibility) is a consequence of (6.4) or (6.7). In the general case, the second equation of the system describes multimass systems $m_i s_i = ms$. Thus, for $s > 1$, it is necessary to consider both equations.

Consider some characteristics of the equations associated with finite-dimensional irreps of the Lorentz group. If we reject the first equation of the system (i.e., the condition of mass irreducibility), then for the second equation of the system, the component $j^0$ of the current vector is positive definite only for $s = 1/2$, and the energy density $-T^{00}$ [see (4.27)] is positive definite only for $s = 1$. [The case $s = 1$ in $3+1$ dimensions has been considered in detail in

Gel'fand *et al.* (1963) and Ghose (1996)]. However, for the system as a whole, the component $j^0$ of the current vector is positive definite for any half-integer spin, and the energy density is positive definite for any integer spin. In the rest frame, half-integer spin solutions with the opposite sign of $p_0$ are characterized by opposite parity.

For the case of infinite-component equations in $2+1$ dimensions, the energy is positive definite for any spin, and $j^0$ is positive or negative definite in accordance with the sign of the charge.

Consideration of the field on the Poincaré group also relates to practical computations for multicomponent equations. As noted in Ginzburg (1956), the general investigation of Gel'fand–Yaglom equations "revealed a number of interesting features, but . . . the use of such equations (or more accurately, systems of a large or infinite number of equations) for any practical computations is not possible." In the present approach, due to the use of spin differential operators instead of finite- or infinite-dimensional matrices, from the technical point of view, there is no essential distinction in the consideration of the equations associated with various finite-dimensional and infinite-dimensional representations of the Lorentz group. Therefore, the present approach can work with higher spins and positive-energy wave equations. For example, the use of spin variables $\mathbf{z}$ has allowed us to obtain an explicit compact form of general plane wave solutions for any spin (including fractional spin in $2+1$ dimensions).

Notice that unlike the equations for particles with unique mass and spin, in the general case, RWE with mass and spin spectrum can either connect several scalar functions $f(x, \mathbf{z})$ (e.g., general Gel'fand–Yaglom equations and, in particular, Bhabha equations) or describe objects with composite spin, which correspond to the functions $f(x, \mathbf{z}_{(1)}, \ldots, \mathbf{z}_{(n)})$ of one set of space-time coordinates $x$ and several sets of spin coordinates $\mathbf{z}$ (e.g., Ivanenko–Landau–Kähler or Dirac–Kähler equation).

## 7. CONCLUSION

In this paper, we elaborated a general scheme of analysis for fields on the Poincaré group and applied it in two-, three-, and four-dimensional cases.

Considering the left GRR of the Poincaré group, we introduced the scalar field $f(x, \mathbf{z})$ on the group, where $x$ are coordinates in Minkowski space and $\mathbf{z}$ are coordinates on the Lorentz group. The connection between the left GRR and the scalar field allows us to use the powerful mathematical method of harmonic analysis on a group, at the same time supporting physical considerations.

Consideration of the functions $f(x, \mathbf{z})$ guarantees the possibility to describe arbitrary spin particles because any irrep of a group is equivalent to some subrepresentation of GRR. Thus, we deal with unique field containing all masses and spins. As a consequence, we have the following points:

1. The explicit form of spin projection operators does not depend on the spin value. These operators are differential operators with respect to $\mathbf{z}$.
2. For this scalar field and thus for arbitrary spin, discrete transformations $C, P, T$ are defined as the automorphisms of the Poincaré group.
3. RWE arise under the classification of the functions on the Poincaré group by eigenvalues of invariant operators and have the same form for arbitrary spin.

The switch to the usual multicomponent description by functions $\psi_n(x)$ corresponds to a separation of the space-time and spin variables, $f(x, \mathbf{z}) = \sum \phi_n(\mathbf{z})$ $\psi_n(x)$, where $\phi_n(\mathbf{z})$ and $\psi_n(x)$ transform under contragradient representations of the Lorentz group. The use of the transformation rules of $x$, $\mathbf{z}$ under automorphisms enables us to deduce the transformation rules of $\psi_n(x)$ under $C, P, T$ without any consideration of the specific form of equations of motion.

We showed that in even dimensions, the consistent consideration of RWE invariant with respect to space reflection requires the use of generators of the group $SO(D, 2)$, which is an extension of the corresponding Lorentz group $SO(D, 1)$.

We gave the interpretation of the right generators belonging to the complete set of commuting operators on the Poincaré group. This interpretation is similar to the Wigner and Casimir interpretation of right generators of the rotation group in the nonrelativistic theory (Biedenharm and Louck, 1981; Wigner, 1959). As in the nonrelativistic case, right generators define quantum numbers that do not depend on the choice of the laboratory frame. In particular, in the $(3 + 1)$-dimensional case, three right generators of the Poincaré group define Lorentz characteristics $j_1$, $j_2$, and chirality, and the fourth right generator distinguishes particles and antiparticles.

Using complete sets of the commuting operators on the group, we classified scalar functions $f(x, \mathbf{z})$. As one of the results of this classification, we reproduced essentially all known finite-component RWE. Moreover, such an approach allowed us to consider alternative possibilities that had not been formulated before. In particular, in the $(3 + 1)$-dimensional case we wrote general subsidiary conditions (5.43) corresponding to $s \neq j_1 + j_2$. On the other hand, instead of subsidiary conditions, one can consider functions of momentum $p$ and spin variables $\mathbf{z}$ with invariant constraints (5.39). We showed that the set of operators related to higher spin equations in $3 + 1$ dimensions obeys commutation relations of $so(3, 3)$ algebra, which coincide with the algebra of $\gamma$-matrices for spin $1/2$. But unlike the latter case, the set of operators for higher spin equations is not closed with respect to anticommutation.

In the framework of the classification of scalar functions, we also get positive energy wave equations allowing a probability amplitude interpretation and associated with infinite-dimensional unitary representations of the Lorentz group.

Along with the alternative description of integer- or half-integer-spin fields, just these equations ensure description of fractional spin fields in $1+1$ and $2+1$ dimensions.

The consideration of the scalar field on the Poincaré group allowed us both to obtain new results and to reproduce the main results of RWE theory, which earlier were obtained by means of different approaches. Thus, a general approach to the construction of different types of RWE is established. One also can consider this as an alternative method to construct a detailed theory of the Poincaré group representations.

The approach under consideration can be directly applied to higher dimensional cases and possibly be generalized to other space-time symmetry groups, such as de Sitter and conformal groups.

## APPENDIX A: BASES OF 2 + 1 LORENTZ GROUP REPRESENTATIONS AND $S^\mu$ MATRICES

Spin projection operators $\hat{S}^\mu$ acting in the space of the functions $f(x, \mathbf{z})$ of $x = (x^\mu)$ and two complex variables $z^1 = z_2$, $z^2 = -z_1$, $|z_1|^2 - |z_2|^2 = |z^2| - |z^1| = 1$ have the form

$$\hat{S}^\mu = \frac{1}{2}(z\gamma^\mu\partial_z - \overset{*}{z}\overset{*}{\gamma}{}^\mu\ \partial_{\overset{*}{z}}), \quad z = (z^1\ z^2),\ \partial_z = (\partial/\partial z^1\ \partial/\partial z^2)^T, \tag{A1}$$

where $\gamma^\mu = (\sigma_3, i\sigma_2, -i\sigma_1)$. For $z = (z_1\ \ z_2)$, the relation $\hat{S}^\mu = -\frac{1}{2}(z\overset{*}{\gamma}{}^\mu\partial_z - \overset{*}{z}\gamma^\mu\partial_{\overset{*}{z}})$ is valid.

The polynomials of the power $2S$ in $z$, which correspond to finite-dimensional irreps $T_S^0$ of the $2+1$ Lorentz group, can be written in the form

$$T_S^0:\quad f_S(x, z) = \sum_{n=0}^{2S} \phi^n(z)\psi_n(x),$$
$$\phi^n(z) = \left(C_{2S}^n\right)^{1/2}(z^1)^{2S-n}(z^2)^n, \quad s^0 = S - n, \tag{A2}$$

where $s^0$ is an eigenvalue of $\hat{S}^0$, and $C_{2S}^n$ are binomial coefficients. The quasipolynomials of the power $2S \leq -1$, which correspond to infinite-dimensional unitary irreps $T_S^\pm$ of the $2+1$ Lorentz group, can be written in the form

$$T_S^+:\quad f_S(x, z) = \sum_{n=0}^{\infty} \phi^n(z)\psi_n(x),$$
$$\phi^n(z) = \left(C_{2S}^n\right)^{1/2}(z^2)^{2S-n}(z^1)^n, \quad s^0 = -S + n,$$
$$T_S^-:\quad f_S(x, \overset{*}{z}) = \sum_{n=0}^{\infty} \phi^n(z)\psi_n(x), \tag{A3}$$

$$\phi^n(z) = \left(C_{2S}^n\right)^{1/2}(\overset{*}{z}{}^{\dot 2})^{2S-n}(\overset{*}{z}{}^{\dot 1})^n, \quad s^0 = S - n,$$
$$C_{2S}^n = \left(\frac{(-1)^n\Gamma(n-2S)}{n!\,\Gamma(-2S)}\right)^{1/2}.$$

There is a correspondence between the action of differential operators $\hat{S}^\mu$ on the functions $f(x, \mathbf{z}) = \phi(\mathbf{z})\psi(x)$ and the multiplication of matrices $\hat{S}^\mu$ by columns $\psi(x)$ composed of $\psi_n(x)$, $\hat{S}^\mu f(x, \mathbf{z}) = \phi(\mathbf{z})S^\mu\psi(x)$. For the finite-dimensional representations $T_S^0$, we have $(S^0)^\dagger = S^0$, $(S^k)^\dagger = -S^k$,

$$(S^{(0)})_n^{n'} = \delta_{nn'}(S-n), \quad n = 0, 1, \ldots, 2S,$$
$$(S^1)_n^{n'} = -\frac{i}{2}\left(\delta_{n,n'+1}\sqrt{(2S-n+1)n} + \delta_{n+1,n'}\sqrt{(2S-n)(n+1)}\right),$$
$$(S^2)_n^{n'} = -\frac{1}{2}\left(\delta_{n,n'+1}\sqrt{(2S-n+1)n} - \delta_{n+1,n'}\sqrt{(2S-n)(n+1)}\right). \quad \text{(A4)}$$

The matrices $S^\mu$ satisfy the condition $(S^\mu)^\dagger = \Gamma S^\mu \Gamma$, where $\Gamma$ is a diagonal matrix, $(\Gamma)_n^{n'} = (-1)^n\delta_{nn'}$. The substitution $z \to \overset{*}{z}$ in (A2) changes only signs of $S^0$ and $S^2$. For representations $T_S^+$ of discrete positive series, we have $(S^\mu)^\dagger = S^\mu$,

$$(S^{(0)})_n^{n'} = \delta_{nn'}(-S+n), \quad n = 0, 1, 2, \ldots,$$
$$(S^1)_n^{n'} = -\frac{1}{2}\left(\delta_{n,n'+1}\sqrt{(n-1-2S)n} + \delta_{n+1,n'}\sqrt{(n-2S)(n+1)}\right),$$
$$(S^2)_n^{n'} = \frac{i}{2}\left(\delta_{n,n'+1}\sqrt{(n-1-2S)n} - \delta_{n+1,n'}\sqrt{(n-2S)(n+1)}\right). \quad \text{(A5)}$$

For $T_S^-$ matrices, $S^1$ has the same form, whereas $S^0$, $S^2$ change only their signs.

The case of representations of principal series which are not bounded by the highest (lowest) weight, was considered in Gitman and Shelepin (1997).

For representations, which correspond to finite-dimensional irreps $T_S^0$, the decomposition (A2) can be written in terms of symmetric spin-tensors $\psi_{\alpha_1\cdots\alpha_{2S}}(x) = \psi_{\alpha_{(1}\cdots\alpha_{2S)}}(x)$,

$$f_S(x, z) = \psi_{\alpha_1\cdots\alpha_{2S}}(x)z^{\alpha_1}\cdots z^{\alpha_{2S}}. \quad \text{(A6)}$$

Comparing the decompositions (A2) and (A6), we obtain the relation

$$\left(C_{2S}^n\right)^{1/2}\psi_n(x) = \psi_{\underbrace{1\cdots1}_{2S-n}\underbrace{2\cdots2}_{n}}(x). \quad \text{(A7)}$$

## APPENDIX B: BASES OF 3 + 2 DE SITTER AND 3 + 1 LORENTZ GROUP REPRESENTATIONS AND $\Gamma^\mu$ MATRICES

Consider polynomials of elements of the Dirac $z$-spinor $Z_{\rm D} = (z^\alpha, \underline{\overset{*}{z}}_{\dot\alpha})$. Any polynomial of power $2S$ can be decomposed in the basis of $(2S+3)!/[6(2S)!]$ monomials

$$(z^1)^a (z^2)^b \, \underline{\overset{*}{z}}{}_1^c \underline{\overset{*}{z}}{}_2^d, \quad a+b+c+d = 2S.$$

We can write 16 operators, which conserve the power of the polynomial:

$$\hat S^{\mu\nu} = \frac{1}{2}\big((\sigma^{\mu\nu})_\alpha^\beta z^\alpha \partial_\beta + (\bar\sigma^{\mu\nu})^{\dot\alpha}_{\dot\beta} \, \underline{\overset{*}{z}}_{\dot\alpha} \, \underline{\partial}^{\dot\beta}\big) - \text{c.c.}, \tag{B1}$$

$$\hat\Gamma^\mu = \hat V^\mu_{12} + \hat V^\mu_{21} - \text{c.c.} = \frac{1}{2}\big(\bar\sigma^{\mu\dot\alpha\alpha} \, \underline{\overset{*}{z}}_{\dot\alpha} \, \partial_\alpha + \sigma^\mu_{\alpha\dot\alpha} z^\alpha \underline{\partial}^{\dot\alpha}\big) - \text{c.c.}, \tag{B2}$$

$$\hat{\underline{\Gamma}}^\mu = i\big(\hat V^\mu_{12} - \hat V^\mu_{21}\big) + \text{c.c.} = \frac{i}{2}\big(\bar\sigma^{\mu\dot\alpha\alpha} \, \underline{\overset{*}{z}}_{\dot\alpha} \, \partial_\alpha - \sigma^\mu_{\alpha\dot\alpha} z^\alpha \underline{\partial}^{\dot\alpha}\big) + \text{c.c.}, \tag{B3}$$

$$\hat\Gamma^5 = \frac{1}{2}(z^\alpha \partial_\alpha - \underline{\overset{*}{z}}_{\dot\alpha} \, \underline{\partial}^{\dot\alpha}) + \text{c.c.}, \tag{B4}$$

$$\hat{\mathcal{T}} = -\hat S_3^{\rm R} = \frac{1}{2}(z^\alpha \partial_\alpha + \underline{\overset{*}{z}}_{\dot\alpha} \, \underline{\partial}^{\dot\alpha}) - \text{c.c.}, \tag{B5}$$

where $\partial_\alpha = \partial/\partial z^\alpha$, $\underline{\partial}^{\dot\alpha} = \partial/\partial \, \underline{\overset{*}{z}}_{\dot\alpha}$,

$$(\sigma^{\mu\nu})_\alpha^\beta = -\frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \bar\sigma^\nu\bar\sigma^\mu)_\alpha^\beta, \quad (\bar\sigma^{\mu\nu})^{\dot\alpha}_{\dot\beta} = -\frac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)^{\dot\alpha}_{\dot\beta}, \tag{B6}$$

and c.c. is the complex conjugate term corresponding to the action in the space of polynomials of the elements of $\underline{Z}_{\rm D} = (\underline{z}^\alpha, \overset{*}{z}_{\dot\alpha})$. The operator $\hat{\mathcal{T}}$ commutes with the other 15 operators and defines the $(\pm)$ power of the polynomials for functions of $Z_{\rm D}$ and $\underline{Z}_{\rm D}$, respectively. Operators (B1)–(B4) obey the commutation relations of $so(3,3) \sim sl(4,R)$ algebra,

$$\begin{aligned}
&[\hat\Gamma^5, \hat S^{\mu\nu}] = 0, \quad [\hat\Gamma^5, \hat\Gamma^\mu] = i\hat{\underline{\Gamma}}^\mu, \quad [\hat\Gamma^5, \hat{\underline{\Gamma}}^\mu] = -i\hat\Gamma^\mu,\\
&[\hat{\underline{\Gamma}}^\mu, \hat{\underline{\Gamma}}^\nu] = -i\hat S^{\mu\nu}, \quad [\hat S^{\lambda\mu}, \hat{\underline{\Gamma}}^\nu] = i(\eta^{\mu\nu}\hat{\underline{\Gamma}}^\lambda - \eta^{\lambda\nu}\hat{\underline{\Gamma}}^\mu),\\
&[\hat\Gamma^\mu, \hat{\underline{\Gamma}}^\nu] = -i\eta^{\mu\nu}\hat\Gamma^5,
\end{aligned} \tag{B7}$$

[see also (5.58), (5.59)]. Using the notations $\hat S^{4\mu} = \hat\Gamma^\mu$, $\hat S^{5\mu} = \hat{\underline{\Gamma}}^\mu$, $\hat S^{54} = \hat\Gamma^5$, we can rewrite the commutation relations in the form (5.62), where $\eta_{55} = \eta_{44} = \eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$. However, for unitary representations of the Poincaré group, all the generators and, in particular, $\hat B_3^{\rm R} = \mp i\hat\Gamma^5$ (for functions of $Z_{\rm D}$ and $\underline{Z}_{\rm D}$, respectively), are Hermitian. Thus, setting $\hat S^{5\mu} = i\hat{\underline{\Gamma}}^\mu$, $\hat S^{54} = i\hat\Gamma^5$, for these representations, it is natural to consider an algebra $so(4, 2) \sim su(2, 2)$ of Hermititan operators.

Supplementing the generators $\hat{S}^{\mu\nu}$ of the Lorentz group by four operators $\hat{\Gamma}^{\mu}$ (or $\hat{\underline{\Gamma}}^{\mu}$), we obtain the algebra of the $3 + 2$ de Sitter group $SO(3, 2)$. Generators in finite-dimensional representations of $SO(3, 2)$ obey the relations $\hat{\Gamma}^{0\dagger} = \hat{\Gamma}^{0}$, $\hat{\Gamma}^{k\dagger} = -\hat{\Gamma}^{k}$.

The linear functions of $z$, $f(x, z) = Z_{\mathrm{D}}\Psi_{\mathrm{D}}(x)$, correspond to the four-dimensional bispinor representation. In the space of columns $\Psi_{\mathrm{D}}(x)$, the operators act as matrices

$$\hat{S}^{\mu\nu} \to \sigma^{\mu\nu}/2, \quad \hat{\Gamma}^{\mu} \to \gamma^{\mu}/2, \quad \hat{\Gamma}^{5} \to \gamma^{5}/2, \quad \hat{\underline{\Gamma}}^{\mu} \to i\gamma^{\mu}\gamma^{5}/2, \quad \hat{\mathcal{T}} \to 1/2. \tag{B8}$$

In accordance with the general theory, Dirac matrices and spin-1 Duffin–Kemmer matrices obey commutation relations of $so(3, 3)$ algebra (Hepner, 1962; Petráš, 1995).

Using (2.61)–(2.63), we get the following for the action of the discrete transformations on the operators (B1)–(B5):

| | $\hat{S}^{\mu\nu}$ | $\hat{\Gamma}^{\mu}$ | $\hat{\underline{\Gamma}}^{\mu}$ | $\hat{\Gamma}^{5}$ | $\hat{\mathcal{T}}$ |
|---|---|---|---|---|---|
| $C$ | $-1$ | $-1$ | $1$ | $1$ | $-1$ |
| $P, T'$ | $(-1)^{\delta_{0\mu}+\delta_{0\nu}}$ | $-(-1)^{\delta_{0\mu}}$ | $(-1)^{\delta_{0\mu}}$ | $-1$ | $1$ |
| $T_{\mathrm{Sch}}$ | $-(-1)^{\delta_{0\mu}+\delta_{0\nu}}$ | $(-1)^{\delta_{0\mu}}$ | $(-1)^{\delta_{0\mu}}$ | $-1$ | $-1$ |

(B9)

It is possible to construct two equations linear in $\hat{p}^{\mu}$ for the scalar functions $f(x, \mathbf{z})$ which are invariant under the proper Poincaré group

$$(\hat{p}_{\mu}\hat{\Gamma}^{\mu} - \kappa) f(x, \mathbf{z}) = 0, \quad (\hat{p}_{\mu}\hat{\underline{\Gamma}}^{\mu} - \kappa) f(x, \mathbf{z}) = 0, \tag{B10}$$

but in accordance with (B9), only the operator $\hat{p}_{\mu}\hat{\Gamma}^{\mu}$ is invariant under space reflection; the operator $\hat{p}_{\mu}\hat{\underline{\Gamma}}^{\mu}$ changes the sign. Thus, only the first equation is invariant under space reflection.

Operators $\hat{\Gamma}^{5}$ and $\hat{p}_{\mu}\hat{\Gamma}^{\mu}$ commute with all the left generators of the Poincaré group, but do not commute with each other, $[\hat{\Gamma}^{5}, \hat{p}_{\mu}\hat{\Gamma}^{\mu}] = i\,\hat{p}_{\mu}\hat{\underline{\Gamma}}^{\mu}$. Therefore, the chirality of a massive particle described by the equation $(\hat{p}_{\mu}\hat{\Gamma}^{\mu} - ms) f(x, \mathbf{z}) = 0$ is uncertain. The operator $\hat{\mathcal{T}}$ commutes both with all left generators of the Poincaré group and with operators $\hat{\Gamma}^{\mu}$; therefore, one may relate to this operator a conserved quantum number changing sign under charge conjugation.

On the polynomials of four complex variables $z^{\alpha}$, $\overset{*}{\underline{z}}_{\dot{\alpha}}$ one can realize symmetric irreps $T_{[2S\,0\,0]}$ of $SL(4, R) \sim SO(3, 3)$. These irreps are a symmetric part of $2S$-times the direct product of fundamental four-dimensional irreps $T_{[100]}$ and remain irreducible after the reduction on the subgroup $SO(3, 2)$, $T_{[2S\,0\,0]} \to T_{[2S0]}$. Here we use notation different from Bhabha (1945): $[2S\ j]$ corresponds to $(j + S\ S)$ in the notation of Bhabha (1945).

We will consider two bases of the finite-dimensional irrep $T_{[2s\,0]}$ of $SO(3,2)$, namely, bases consisting of eigenfunctions of the operator $\hat{\Gamma}^5$ or $\hat{\Gamma}^0$. The first basis corresponds to the chiral representation,

$$\varphi^{m_1 m_2}_{j_1 j_2}(z, \overset{*}{\underline{z}}) = N^{1/2}(z^1)^{j_1+m_1}(z^2)^{j_1-m_1}\, \overset{*}{\underline{z}}{}_{\dot 1}^{j_2+m_2}\, \overset{*}{\underline{z}}{}_{\dot 2}^{j_2+m_2}, \tag{B11}$$

where $s = j_1 + j_2$, $\lambda = j_1 - j_2$, $m_1$ and $m_2$ are eigenvalues of the operators $\hat{M}_3$ and $\hat{N}_3$, which are linear combinations of $\hat{S}_3$ and $\hat{B}_3$ [see (5.13)], $N = (2s)!/[(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!]$. Consisting of eigenfunctions of the $\hat{\Gamma}^0$ basis,

$$\phi^{n_1 n_2}_{k_1 k_2}(z, \overset{*}{\underline{z}}) = (N')^{1/2}(z^1 + \overset{*}{\underline{z}}_{\dot 1})^{k_1+n_1}(z^2 + \overset{*}{\underline{z}}_{\dot 2})^{k_1-n_1}(z^1 - \overset{*}{\underline{z}}_{\dot 1})^{k_2+n_2}(z^2 - \overset{*}{\underline{z}}_{\dot 2})^{k_2-n_2}, \tag{B12}$$

where $s = k_1 + k_2$ and $N' = (2s)!/[(k_1+n_1)!(k_1-n_1)!(k_2+n_2)!(k_2-n_2)!]$, corresponds for $s = 1/2$ to the Dirac representation. The functions (B12) are eigenfunctions of the operators $\hat{\Gamma}^0$, $\underline{\hat{\Gamma}}^3$, $\hat{S}_3$ with eigenvalues $k_1 - k_2$, $i(n_1 - n_2)/2$, $(n_1 + n_2)/2$, respectively. For fixed $s$, we have

$$\begin{aligned} f_s(x, z, \overset{*}{\underline{z}}) &= \sum_{j_1+j_2=s}\sum_{m_1,m_2} \psi^{m_1 m_2}_{j_1 j_2}(x)\varphi^{m_1 m_2}_{j_1 j_2}(z, \overset{*}{\underline{z}}) \\ &= \sum_{k_1+k_2=s}\sum_{n_1,n_2} \psi^{n_1 n_2}_{k_1 k_2}(x)\phi^{n_1 n_2}_{k_1 k_2}(z, \overset{*}{\underline{z}}). \end{aligned} \tag{B13}$$

Below, we will use the basis (B12). According to (5.74), in the rest frame for a particle described by the system (5.71)–(5.73), we have

$$\begin{aligned} f(x, z, \overset{*}{\underline{z}}) &= \psi^+(x)\phi^+_{s,s^3}(z, \overset{*}{\underline{z}}) + \psi^-(x)\phi^-_{s,s^3}(z, \overset{*}{\underline{z}}) \\ &= C_1 e^{imx^0}\phi^+_{s,s^3}(z, \overset{*}{\underline{z}}) + C_2 e^{-imx^0}\phi^-_{s,s^3}(z, \overset{*}{\underline{z}}), \\ \phi^+_{s,s^3}(z, \overset{*}{\underline{z}}) &= \left(z^1 + \overset{*}{\underline{z}}_{\dot 1}\right)^{s+s^3}\left(z^2 + \overset{*}{\underline{z}}_{\dot 2}\right)^{s-s^3}, \\ \phi^-_{s,s^3}(z, \overset{*}{\underline{z}}) &= \left(z^1 - \overset{*}{\underline{z}}_{\dot 1}\right)^{s+s^3}\left(z^2 - \overset{*}{\underline{z}}_{\dot 2}\right)^{s-s^3}. \end{aligned} \tag{B14}$$

The equation $(\hat{p}_\mu \hat{\Gamma}^\mu - sm) f(x, z, \overset{*}{\underline{z}}) = 0$ has the matrix form

$$(\hat{p}_\mu \Gamma^\mu - sm)\psi(x) = 0, \tag{B15}$$

where $\psi(x)$ is a column. It is convenient to enumerate the basis elements (B12) [and the elements of the column $\psi(x)$] in order of decrease of $k_1 - k_2 = s, s-1, \ldots, -s$. Matrices $\Gamma^\mu$ obey the relations $\Gamma^{0\dagger} = \Gamma^0$, $\Gamma^{k\dagger} = -\Gamma^k$. Matrix $\Gamma^0$ is diagonal and has the elements $k_1 - k_2$. Matrices $\Gamma^1$ and $\Gamma^3$ are skew-symmetric real, and $\Gamma^2$ is symmetric imaginary. According to (B2), the matrices $\Gamma^k$ have nonzero elements only in blocks corresponding to the transitions $(k_1, k_2) \to (k_1 \pm 1/2, k_2 \mp 1/2)$. Using this property, it is easy to see that the diagonal matrix $\Gamma$ with the elements $(-1)^{2k_2}$ commutes with $\Gamma^0$ and anticommutes with $\Gamma^k_{\leftrightarrow}$, $\Gamma^{\mu\dagger} = \Gamma\Gamma^\mu\Gamma$. This allows one to rewrite the Hermitian-conjugate equation $\psi^\dagger(\hat{p}_\mu \; \Gamma^{\mu\dagger} + sm) = 0$ in the form

$$\bar{\psi}(x)(\overleftarrow{\hat{p}}_\mu \; \Gamma^\mu + sm) = 0, \quad \bar{\psi} = \psi^\dagger\Gamma, \tag{B16}$$

and to define the invariant scalar product in the space of columns as $\int \bar{\psi}(x)\psi(x) \, d^3x$. As a consequence of (B15) and (B16), the continuity equation holds,

$$\partial_\mu j^\mu = 0, \quad j^\mu = \bar{\psi}\Gamma^\mu\psi.$$

Now the question concerning the positive definiteness of the current vector component $j^0$ and the energy density may be consider similarly to the $(2+1)$-dimensional case (see Section 3). For half-integer-spin particles described by the system (5.71)–(5.73), the charge density $j^0$ is positive definite, since in the rest frame [see (B14)], $j^0 = \bar{\psi}\Gamma\Gamma^0\psi = s(|\psi^+(x)| + |\psi^-(x)|) > 0$. The energy density [defined in terms of the energy-momentum tensor (4.27)] and the scalar product $\bar{\psi}\psi$ are indefinite since in the rest frame, they are proportional to $|\psi^+(x)| - |\psi^-(x)|$. For integer-spin particles, the energy density is positive definite, and the scalar product and $j^0$ are indefinite.

Consider discrete transformations in terms of the columns $\psi(x)$. According to (5.8), under space reflection, $\phi^{n_1 n_2}_{k_1 k_2}(z, \overset{*}{\underline{z}}) \to (-1)^{2k_1}\phi^{n_1 n_2}_{k_1 k_2}(z, \overset{*}{\underline{z}})$. Whence, taking into account $f(x, z, \overset{*}{\underline{z}}) \to f(x', z') = \phi(z, \overset{*}{\underline{z}})\psi'(x')$, we get

$$\psi(x) \overset{P}{\to} (-1)^{2s}\Gamma\psi(\bar{x}), \quad \text{where } \bar{x} = (x^0, -x^k). \tag{B17}$$

According to (2.63), under charge conjugation, $\phi^{n_1 n_2}_{k_1 k_2}(z, \overset{*}{\underline{z}}) \to \phi^{n_1 n_2}_{k_1 k_2}(\overset{*}{z}, \underline{z})$. Taking into account that $\phi^{n_1 n_2}_{k_1 k_2}(\overset{*}{z}, \underline{z})$ and $(-1)^{s+n_1-n_2}\phi^{n_2 n_1}_{k_2 k_1}(z, \overset{*}{\underline{z}})$ have the same transformation rule, we get

$$\psi^{n_1 n_2}_{k_1 k_2}(x) \overset{C}{\to} (-1)^{s+n_1-n_2}\overset{*}{\psi}{}^{n_2 n_1}_{k_2 k_1}(x). \tag{B18}$$

In particular, for $s = 1/2$, using the relation $f(x, z, \overset{*}{\underline{z}}) = Z_{\rm D}\Psi(x)$, we get $\Psi(x) \overset{P}{\to} \gamma^0\Psi(\bar{x})$, $\Psi(x) \overset{C}{\to} \Psi^c(x) = C\bar{\psi}^T(x)$, where $C$ is the matrix with elements $-i\sigma_2$ on secondary diagonal, $C = i\gamma^2\gamma^0$. The transformation properties of the bilinear $\bar{\psi}\Gamma^\mu\psi$, $\bar{\psi}\Gamma^5\psi$, $\bar{\psi}\underline{\Gamma}^\mu\psi$ under $C, P, T$ coincide with those of the corresponding operators [see (B9)].

## ACKNOWLEDGMENTS

This work was partially supported by Brazilian Agencies CNPq (D.M.G.) and FAPESP (D.M.G., A.L.Sh.). The authors would like to thank I. L. Buchbinder, L. A. Shelepin, S. N. Solodukhin, I. V. Tyutin, and A. A. Deriglazov for useful discussions.

## REFERENCES

Abdalla, E., Abdalla, M. C. B., and Rothe, K. D. (1991). *2 Dimensional Quantum Field Theory*, World Scientific, Singapore.

Ahluwalia, D. V. and Ernst, D. J. (1992). Paradoxical kinematic acasuality in Weinberg's equations for massless particles of arbitrary spin. *Mod. Phys. Lett. A* **7**, 1967–1974.

Ahluwalia, D. V., Jonson, M. B., and Goldman, T. (1993). A Bargmann–Wightman–Wigner-type quantum field theory. *Phys. Lett. B* **316**, 102–108.

Aragone, C. and Deser, S. (1984). Topologically massive spin-5/2 theory. *Class. Quantum Grav.* **1**, 331–337.

Aragone, C., Deser, S., and Yang, Z. (1987). *Massive higher spin from dimensional reduction of gauge fields. Ann. Phys.* (N. Y.) **179**, 76–96.

Arodź, H. (1976). Metric tensors, Lagrangian formalism and Abelian gauge field on the Poincaré group. *Acta Phys. Pol. B* **7**, 177–190.

Bacry, H. and Kihlberg, A. (1969). Wavefunctions on homogeneous spaces. *J. Math. Phys.* **10**, 2132–2141.

Bargmann V. and Wigner, E. P. (1948). Group theoretical discussion of relativistic wave equations. *Proc. Natl. Acad. Sci. USA* **34**, 211–223.

Barut A. O. and Raczka, R. (1977). *Theory of Group Representations and Applications* PWN, Warsaw.

Benn, I. M. and Tucker, R. W. (1983). Fermions without spinors. *Commun. Math. Phys.* **89**, 341–362.

Benn, I. M. and Tucker, R. W. (1988). *An Introduction to Spinors and Geometry with Applications in Physics*, Adam Hilger, Bristol.

Berezin, F. A. (1966). *The Method of Second Quantization*, Academic, New York.

Bhabha, H. J. (1945). Relativistic wave equations for the elementary particles. *Rev. Mod. Phys.* **17**, 200–216.

Biedenharn, L. C., Braden, H. W., Truini, P., and Van Dam, H. (1988). Relativistic wavefunctions on spinor spaces. *J. Phys. A* **21**, 3593–3610.

Biedenharn, L. S. and Louck, J. D. (1981). *Angular Momentum in Quantum Physics*, Addison-Wesley, Reading, Massachusetts.

Binegar, B. (1982). Relativistic field theories in three dimensions. *J. Math. Phys.* **23**, 1511–1517.

Biritz, H. (1979). Consistent wave equations for families of massive particles with any spin. *Int. J. Theor. Phys.* **18**, 601–688.

Boyer, C. P. and Fleming, G. N. (1974). Quantum field theory on a seven-dimensional homogeneous space of the Poincaré group. *J. Math. Phys.* **15**, 1007–1024.

Briegel, H.-J., Englert, B.-G., and Süssmann, G. (1991). Canonical quantization of the classical Hamiltonian for a relativistic spin-0 particle. *Z. Naturforsch. A* **46**, 933–938.

Buchbinder, I. L., Gitman, D. M., Krykhtin, V. A., and Pershin, V. D. (2000a). Equations of motion of massive spin 2 field coupled to gravity. *Nucl. Phys. B* **584**, 615–640.

Buchbinder, I. L., Gitman, D. M., and Shelepin, A. L. (2000b). Discrete symmetries as automorphisms of the proper Poincaré group. hep-th/0010035.

Buchbinder, I. L., Krykhtin, V. A., and Pershin, V. D. (1999). On consistent equations for massive spine-2 field coupled to gravity in string theory. *Phys. Lett. B* **466**, 216–226.

Buchbinder, I. L. and Kuzenko, S. M. (1995). *Ideas and Methods of Supersymmetry and Supergravity*, IOP, Bristol.

Buchbinder, I. L. and Shvartsman, Sh. M. (1993). Derivation of the actions for the relativistic particles with arbitrary spins. *Int. J. Mod. Phys. A* **8**, 683–703.

Bullinaria, J. A. (1986). Kähler fermions in arbitrary space-times, their dimensional reduction, and relation to spinorial fermions. *Ann. Phys.* (N. Y.) **168**, 301–343.

Capri, A. Z. and Kobes, R. L. (1980). Further problems in spin-3/2 field theories. *Phys. Rev. D* **22**, 1967–1978.

Castell, L. (1967). Linear field equations for particles with arbitrary spin. *Nuovo Cimento A* **50**, 945–951.

Cortés J. L. and Plyushchay, M. S. (1994). Linear differential equations for a fractional spin field. *J. Math. Phys.* **35**, 6049–6057.

Cortes, J. L. and Plyushchay, M. S. (1996). Anyons as spinning particles. *Int. J. Mod. Phys. A* **11**, 3331–3362.

Deriglazov, A. A. and Gitman, D. M. (1999). Classical description of spinning degrees of freedom of relativistic particles by means of commuting spinors. *Mod. Phys. Lett. A* **14**, 709–720.

Deser, S. (1984). Massive spin 3/2 theories in 3 dimensions. *Phys. Lett. B* **140**, 321–323.

Deser, S. and Jackiw, R. (1984). "Self-duality" of topologically massive gauge theories. *Phys. Lett. B* **139**, 371–373.

Deser, S., Jackiw, R., and Templeton, S. (1982). Topologically massive gauge theories. *Ann. Phys.* (N. Y.) **140**, 372–411.

Deser, S. and Kay, J. H. (1983). Topologically massive supergravity. *Phys. Lett. B* **120**, 97–100.

Dirac, P. A. M. (1936). Relativistic wave equations. *Proc. R. Soc. Lond. A* **155**, 447–459.

Dirac, P. A. M. (1971). A positive-energy relativistic wave equation. *Proc. R. Soc. Lond. A* **322**, 435.

Dirac, P. A. M. (1972a). A positive-energy relativistic wave equation. II. *Proc. R. Soc. Lond. A* **328**, 1–7.

Dirac, P. A. M. (1972b). Relativity and quantum mechanics. *Fields Quanta* **3**, 139–164.

Drechsler, W. (1997). Geometro-stohastically quantized fields with internal spin variables. *J. Math. Phys.* **38**, 5531–5558.

Feynman, R. P. and Gell-Mann, M. (1958). Theory of Fermi interaction. *Phys. Rev.* **109**, 193–198.

Fierz, M. and Pauli, W. (1939). On relativistic wave equations for particles of arbitrary spin in an electromagnetic field. *Proc. R. Soc. Lond. A* **173A**, 211–232.

Finkelstein, D. (1955). Internal structure of spinning particles. *Phys. Rev.* **100**, 924–931.

Fradkin, D. M. (1966). Comments on a paper by Majorana concerning elementary particles. *Am. J. Phys.* **34**, 314–318.

Gaioli, F. H. and Alvarez, E. T. G. (1995). Some remarks about intrinsic parity in Ryder's derivation of the Dirac equation. *Am. J. Phys.* **63**, 177–178.

Gavrilov, S. P. and Gitman, D. M. (2000). Quantization of point-like particles and consistent relativistic quantum mechanics. *Int. J. Mod. Phys. A* **15**, 4499–4538.

Gel'fand, I. M., Graev, M. I., and Vilenkin, N. Ya. (1966). *Generalized Functions*, Vol. 5, Academic Press, New York.

Gel'fand, I. M. and Yaglom, A. M. (1948). General relativistic-invariant equations and infinite-dimensional representations of the Lorentz group. *Zh. Ehksp. Teor. Fiz.* **18**, 703–733.

Gel'fand, I. M., Minlos, R. A., and Shapiro, Z. Ya. (1963). *Representations of the Rotation and Lorentz Groups and Their Applications*, Pergamon Press, Oxford.

Ghose, P. (1996). Relativistic quantum mechanics of spin-0 and spin-1 bosons. *Found. Phys.* **26**, 1441–1455.

Ginzburg, V. L. (1956). On relativistic wave equations with a mass spectrum. *Acta Phys. Pol.* **15**, 163–175.

Ginzburg, V. L. and Tamm, I. E. (1947). On the theory of spin. *Zh. Eksp. Teor. Fiz.* **17**, 227–237.

Gitman, D. M. and Shelepin, A. L. (1997). Poincaré group and relativistic wave equations in $2+1$ dimensions. *J. Phys. A* **30**, 6093–6121.

Gitman, D. M. and Shelepin, A. L. (1998). Representations of SU($N$) groups on the polynomials of anticommuting variables. *Kratk. Soob. Fiz. (Lebedev Inst.)* **11**, 21–30.

Gitman, D. M. and Tyutin, I. V. (1997). Pseudoclassical description of higher spins in $2+1$ dimensions. *Int. J. Mod. Phys. A* **12**, 535–556.

Greiner, W. (1997). *Relativistic Quantum Mechanics. Wave Equations*, 2nd Ed., Springer-Verlag, Berlin.

Hai, N. X. (1969). Harmonic analysis on the Poincaré group, I. Generalized matrix elements. *Commun. Math. Phys.* **12**, 331–350.

Hai, N. X. (1971). Harmonic analysis on the Poincaré group, II. The Fourier transform. *Commun. Math. Phys.* **22**, 301–320.

Hamermesh, M. (1960). Galilean invariance and the Srödinger equation. *Ann. Phys.* (N. Y.) **9**, 518–521.

Harish-Chandra. (1947). On relativistic wave equations. *Phys. Rev.* **71**, 793–805.

Hasiewicz, Z. and Siemion, P. (1992). A bosonic model for particles with arbitrary spin. *Int. J. Mod. Phys. A* **7**, 3979–3996.

Hepner, W. A. (1962). The inhomogeneous Lorentz group and the conformal group. *Nuovo Cimento* **26**, 351–368.

Hurley, W. J. (1971). Relativistic wave equations for particles with arbitrary spin. *Phys. Rev. D* **4**, 3605–3616.

Hurley, W. J. (1974). Invariant bilinear forms and the discrete symmetries for relativistic arbitrary-spin fields. *Phys. Rev. D* **10**, 1185–1200.

Ionesco-Pallas, N. J. (1967). Relativistic Schrödinger equation for a particle with arbitrary spin. *J. Franklin Inst.* **284**, 243–250.

Ivanenko, D. and Landau, L. (1928). Zur theorie des magnetischen electrons. I. *Z. Phys.* **48**, 340–348.

Ivanenko, D. D., Obukhov, Yu. N., and Solodukhin, S. N. (1985). On antisymmetric tensor representation of the Dirac equation. Preprint IC/85/2, ICTP, Trieste.

Jackiw, R. and Nair, V. P. (1991). Relativistic wave equation for anyons. *Phys. Rev. D* **43**, 1933–1942.

Joos, H. (1962). Zur darstellugstheorie der inhomogenen Lorentzgrouppe als grundlade quantenmechanische kinematick. *Fortschr. Phys.* **10**, 65.

Kähler, E. (1962). Der innere differentialkalkül. *Rend. Math.* **21**, 425.

Kemmer, N. (1939). The particle aspect of meson theory. *Proc. R. Soc. Lond. A* **173**, 91–116.

Kemmer, N., Polkinghorn, J. C., and Pursey, D. L. (1959). Invariance in elementary particle physics. *Rep. Progr. Phys.* **22**, 368–432.

Kihlberg, A. (1964). On the internal degrees of freedom of elementary particles. *Ark. Fys.* **28**, 121–127.

Kihlberg, A. (1970). Fields on a homogeneous space of the Poincare group. *Ann. Inst. Henri Poincaré* **13**, 57–76.

Kim, Y. S. and Noz, M. E. (1986). *Theory and Applications of the Poincaré Group*, Reidel, Dordrecht.

Kim, Y. S. and Wigner, E. P. (1987). Cylindrical group and massless particles. *J. Math. Phys.* **28**, 1175–1179.

Klishevich, S. M. (2000). Massive fields with arbitrary integer spin in homogeneous electromagnetic field. *Int. J. Mod. Phys. A* **15**, 535–552.

Krajcik, R. A. and Nieto, M. M. (1976). Bhabha first-order wave equations. IV. Causality with minimal electromagnetic coupling. *Phys. Rev. D* **13**, 924–941.

Krajcik, R. A. and Nieto, M. M. (1977). Bhabha first-order wave equations. VII. Summary and conclusions. *Phys. Rev. D* **15**, 445–452.

Kruglov, S. I. (1999). Pair production and solutions of the wave equation for particles with arbitrary spin. hep-ph/9908410.

Kuzenko, S. M., Lyakhovich, S. L., and Segal, A. Yu. (1995). A geometric model of the arbitrary spin massive particle. *Int. J. Mod. Phys. A* **10**, 1529–1552.

Landau, L. D. and Lifschitz, E. M. (1977). *Quantum Mechanics, volume 3 of Course of Theoretical Physics*, Pergamon, Oxford.

Levy-Leblond, J.-M. (1963). Galilei group and nonrelativistic quantum mechanics. *J. Math. Phys.* **4**, 776–788.

Loide, R.-K., Ots, I., and Saar, R. (1997). Bhabha relativistic wave equations. *J. Phys. A* **30**, 4005–4017.

Lubanski, J. K. (1942). Sur la théorie des particules élémentaires de spin quelconque. *Physica* **9**, 310–338.

Lurçat, F. (1964). Quantum field theory and the dynamical role of spin. *Physics* **1**, 95.

L'vov, D. V., Shelepin, A. L., and Shelepin, L. A. (1994). Computational formalism for the group of motion of a pseudo-euclidean plane. *Phys. Atom. Nucl.* **57**, 1083–1088.

Lyakhovich, S. L., Segal, A. Yu., and Sharapov, A. A. (1996). Universal model of a $D = 4$ spining particle. *Phys. Rev. D* **54**, 5223–5238.

Mackey, G. (1968). *Induced Representations of Groups and Quantum Mechanics*, Benjamin, New York.

Majorana, E. (1932). Teoria relativistica di particelle con momento intrinseco arbitrario. *Nuovo Cimento* **9**, 335–344.

Obukhov, Yu. N. and Solodukhin, S. N. (1993). Reduction of the dirac equation and its connection with the Ivanenko–Landau–Kähler equation. *Theor. Math. Phys.* **94**, 198–210.

Ohnuki, Y. (1988). *Unitary Representations of the Poincaré Group and Relativistic Wave Equations*, World Scientific, Singapore.

Petráš, M. (1995). The *so*(3, 3) group as a common basis for Dirac's and Proca's equations. *Czech. J. Phys. B* **45**, 455–464.

Plyushchay, M. S. (1991). The model of a relativistic particle with torsion. *Nucl. Phys. B* **362**, 54–72.

Plyushchay, M. S. (1992). The model of a relativistic particle with fractional spin. *Int. J. Mod. Phys. A* **7**, 7045–7064.

Plyushchay, M. S. (1997). R deformed Heisenberg algebra, anyons and $d = (2 + 1)$ supersymmetry. *Mod. Phys. Lett. A* **12**, 1153–1164.

Rarita, W. and Schwinger, J. (1941). On a theory of particles with half-integral spin. *Phys. Rev.* **60**, 61.

Rideau, G. (1966). On the reduction of the regular representation of the Poincaré group. *Commun. Math. Phys.* **3**, 218–227.

Ryder, L. (1988). *Quantum Field Theory*, Cambridge University, Cambridge.

Samarov, K. L. (1984). On the pseudodifferential Schrödinger equation. *Sov. Phys. Dokl.* **29**, 909–910.

Schwinger, J. (1951). The theory of quantized fields. I. *Phys. Rev.* **82**, 914–927.

Shelepin, A. L. (1997). Processes with jumps and pseudodifferential Schrödinger and Fokker–Planck equations. *Phys. Atom. Nucl.* **60**, 206–217.

Shirokov, Yu. M. (1951). Relativistskaia teoria spina. *Zh. Eksp. Teor. Fiz.* **21**, 748–760.

Singh, L. P. S. and Hagen, C. R. (1974a). Lagrangian formulation for arbitrary spin. I. The boson case. *Phys. Rev. D* **9**, 898–909.

Singh, L. P. S. and Hagen, C. R. (1974b). Lagrangian formulation for arbitrary spin. II. The fermion case. *Phys. Rev. D* **9**, 910–920.

Smith, J. R. (1993). Second quantization of the square-root Klein–Gordon operator, microscopic causality, propagators, and interaction. Preprint UCD/II RPA 93-13, University of California, Davis.

Stoyanov, D. Tz. and Todorov, I. T. (1968). Majorana representations of the Lorentz group and infinite-component fields. *J. Math. Phys.* **9**, 2146–2167.

Sucher, J. (1963). Relativistic invariance and the square-root Klein-Gordon equation. *J. Math. Phys.* **4**, 17–23.

Toller, M. (1978). Classical field theory in the space of reference frames. *Nuovo Cimento B* **44**, 67–98.

Toller, M. (1996). Free quantum fields on the Poincaré group. *J. Math. Phys.* **37**, 2694–2730.

Townsend, P. T., Pilch, K., and Van Nieuwenhuizen, P. (1984). Self-duality in odd dimensions. *Phys. Lett. B* **136**, 38–42.

Tung, Wu-Ki. (1967). Relativistic wave equations and field theory for arbitrary spin. *Phys. Rev.* **156**, 1385–1398.

Tung, W.-K. (1985). *Group Theory in Physics*, World Scientific, Singapore.

Umezava, H., Kamefuchi, S., and Tanaka, S. (1954). On the time reversal in the quantized field theory. *Prog. Theor. Phys.* **12**, 383–400.

Velo, G. (1972). Anomalous behavior of a massiv spin two charged particle in an external electromagnetic field. *Nucl. Phys. B* **43**, 389–401.

Velo, G. and Zwanziger, D. (1969). Propagation and quantization of Rarita–Schwinger waves in an external electromagnetic potential. *Phys. Rev.* **186**, 1337–1341.

Vasiliev, M. A. (1992). More on equations of motion for interacting massless fields of all spins in $3 + 1$ dimensions. *Phys. Lett. B* **285**, 225–234.

Vasiliev, M. A. (1996). Higher-spin gauge theories in four, three and two dimensions. *Int. J. Mod. Phys. D* **5**, 763–797.

Vasiliev, M. A. and Tyutin, I. V. (1997). Lagrangian formulation of ireductible massive fields of arbitrary spin in $(2 + 1)$ dimensions. *Teor. Mat. Fiz.* **113**, 1244–1254.

Vilenkin, N. Ya. (1968). *Special Functions and the Theory of Group Representations*, AMS, Providence, Rhode Island.

Vilenkin, N. Ja. and Klimyk, A. U. (1991). *Representations of Lie Groups and Special Functions,* vol. 1, Kluwer, Dordrecht.

Vshivtsev, A. S., Magnitskii, B. V., Zhukovskii, V. Ch., and Klimenko, K. G. (1998). Dynamical effects in $(2 + 1)$-dimensional theories with four-fermion interaction. *Phys. Part. Nucl.* **29**, 523–547.

Weinberg, S. (1964). Feynman rules for any spin. *Phys. Rev.* **133B**, 1318–1332.

Weinberg, S. (1969). Feynman rules for any spin. III. *Phys. Rev.* **181**, 1893–1899.

Wightman, A. S. (1978). Invariant wave equations: General theory and applications to the external field problem. In *Invariant Wave Equations,* G. Velo and A. S. Wightman, eds. volume 73 of *Lecture Notes in Physics*, Springer-Verlag, Berlin, pp. 1–101.

Wigner, E. P. (1939). On unitary representations of the inhomogeneous Lorentz group. *Ann. Math.* **40**, 149–204.

Wigner, E. P. (1959). *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*, Academic Press, New York.

Wigner, E. P. (1963). In *Theoretical Physics*, A. Salam, ed., IAEA, Trieste, p. 59.

Wybourne B. G. (1974). *Classical Groups for Physicists,* Wiley, New York.

Yukawa, H. (1950). Quantum theory of non-local fields. I. Free fields. *Phys. Rev.* **77**, 219–226.

Zhelobenko, D. P. and Schtern, A. I. (1983). *Representations of Lie Groups,* Nauka, Moscow.

Zwanziger, D. (1978). Method of characteristics in the external field problem or how to recognize an acasual equation when you see one. In *Invariant Wave Equations,* G. Velo and A. S. Wightman, eds. volume 73 of Lecture Notes in Physics, Springer-Verlag, Berlin, pp. 143–164.